Transverse flows in rapidly oscillating cylindrical vessels

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The Starling Resistor

Originally developed by physiologists to study the behaviour of collapsible tubes in physiology:

- Blood flow in veins and arteries
- Flow of air in the airways of the lung
- ...

System readily develops large-amplitude, self-excited oscillations.
Governing Equations

- Unsteady Navier–Stokes equations

\[
\rho \left( \frac{\partial u^*_i}{\partial t^*} + u^*_j \frac{\partial u^*_i}{\partial x^*_j} \right) = - \frac{\partial p^*}{\partial x^*_i} + \mu \frac{\partial}{\partial x^*_i} \left( \frac{\partial u^*_i}{\partial x^*_j} + \frac{\partial u^*_j}{\partial x^*_i} \right), \quad \frac{\partial u^*_i}{\partial x^*_i} = 0
\]

- Coupled to Kirchhoff–Love, thin-shell theory

\[
\int_0^{2\pi a^*} \int_0^L E^{*\alpha\beta\gamma\delta} \left( \gamma^{\alpha\beta\delta\gamma\delta} + \frac{h^2}{12} \kappa^{*\alpha\beta\delta\kappa^{*\gamma\delta}} \right) \, d\xi^*1 \, d\xi^*2 =
\]  
\[
\int_0^{2\pi a} \int_0^L \left( f^* - \rho_w h \frac{\partial^2 R^*_{w}}{\partial t^*} \right) \cdot \delta R^*_{w} \sqrt{A} \, d\xi^*1 \, d\xi^*2.
\]

formulated in Lagrangian coordinates \((\xi^*1, \xi^*2)\).
Governing Equations

- Unsteady Navier–Stokes equations

\[ \rho \left( \frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*} \right) = - \frac{\partial p^*}{\partial x_i^*} + \mu \frac{\partial}{\partial x_i^*} \left( \frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} \right), \quad \frac{\partial u_i^*}{\partial x_i^*} = 0 \]

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\[
\int_0^{2\pi a^*} \int_0^L \left( f^* - \rho_w h \frac{\partial^2 R_{w^*}}{\partial t^*} \right) \cdot \delta R_{w^*} \sqrt{A} \, d\xi^1 \, d\xi^2.
\]

formulated in Lagrangian coordinates \((\xi^1, \xi^2)\).

Strategy:

- Use scaling to simplify equations in a previously unexplored parameter regime.
- Combined numerical/asymptotic study of the system in this regime.
Parameter regime [c.f. Jensen & Heil JFM 481 (2003)]

- Small amplitude of wall oscillation

\[ \epsilon \ll 1. \]

- High frequency of wall oscillation (period \( T \)):

\[ St = \frac{a/T}{U} = \frac{\text{Velocity induced by wall motion}}{\text{Steady velocity}} \gg 1 \]

- Large Reynolds number of the steady through flow

\[ Re = \frac{a\rho U}{\mu} = \frac{\text{Fluid inertia}}{\text{Fluid viscosity}} \gg 1 \]

- Large Womersley number

\[ \alpha^2 = \frac{\rho a^2}{\mu T} = \frac{\text{Timescale for diffusion of vorticity}}{\text{Period of oscillation}} \gg 1 \]
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Such that

\[ \alpha^2 \gg St \sim \frac{1}{\epsilon} \gg \alpha^{3/2} \]
Scaling

- Decompose all quantities into steady and unsteady components:

\[
\mathbf{u} = \mathbf{\bar{u}}(x_j) + \mathbf{\hat{u}}(x_j, t),
\]

\[
p = \bar{p}(x_j) + \hat{p}(x_j, t),
\]
Scaling

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• Expand everything in powers of \( \epsilon \):

\[ \hat{u}(x_j, t) = \hat{u}_0(x_j, t) + \epsilon \hat{u}_1(x_j, t) + \ldots \quad \text{etc.} \]
Scaling

- Decompose all quantities into steady and unsteady components:

\[ u = \overline{u}(x_j) + \hat{u}(x_j, t), \]

\[ p = \overline{p}(x_j) + \hat{p}(x_j, t), \]

- Expand everything in powers of \( \epsilon \):

\( \hat{u}(x_j, t) = \hat{u}_0(x_j, t) + \epsilon \hat{u}_1(x_j, t) + \ldots \) etc.

- Leading-order oscillatory flow \((\hat{u}_0, \hat{p}_0)\) is governed by

\[
\frac{\partial \hat{u}_0}{\partial t} = -\nabla \hat{p}_0 + \left[ \frac{1}{\alpha^2} \nabla^2 \hat{u}_0 \right] \quad \text{and} \quad \nabla \cdot \hat{u}_0 = 0,
\]

(linearised Euler equations with Stokes boundary layers), subject to

\[ \hat{u}_0 = \frac{\partial v_w(t)}{\partial t} \quad \text{on the wall} \]
Scaling

- Decompose all quantities into steady and unsteady components:
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  \]

- **Note:** Leading-order unsteady flow is independent of the steady through flow.
Fluid-structure interaction

- Wall deforms in response to fluid traction: What is the period of the oscillation?

Fluid inertia \( \sim \) Wall elasticity (\( \sim \) bending stiffness)

\[ \rho \left( \frac{a}{T} \right)^2 \sim K \]
**Fluid-structure interaction**

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- \( \implies \) Timescale of oscillation:

  \[
  T = a \sqrt{\frac{\rho}{K}}
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- Timescale of oscillation:

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- Womersley number:

\[ \alpha = \left( \frac{a}{\mu} \right)^{1/2} (K \rho)^{1/4} \]

- Strouhal number:

\[ St = \frac{1}{U} \left( \frac{K}{\rho} \right)^{1/2} \]

Note: \( \alpha \) is a material parameter.
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T = \alpha \sqrt{\frac{\rho}{K}}
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- \( \implies \) Womersley number: Strouhal number:

\[
\alpha = \left( \frac{a}{\mu} \right)^{1/2} (K \rho)^{1/4}
\]
\[
St = \frac{1}{U} \left( \frac{K}{\rho} \right)^{1/2}
\]

Note: \( \alpha \) is a material parameter.

- \( \alpha, St \gg 1 \) can be realised by having relatively stiff walls (large \( K \)).
The wall displacement field

The wall performs small-amplitude non-axisymmetric oscillations about the axisymmetric wall shape

\[ A = A_{\text{undef}} + O(2^2) \]
The wall displacement field

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Observation:
- The change in the tube’s cross-sectional area is a second-order effect:

\[ A = A_{undef} + O(\varepsilon^2). \]
**The wall displacement field**

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**Observation:**
- The change in the tube’s cross-sectional area is a second-order effect:
- For small wall displacements, the oscillatory wall motion does not drive any (net) axial flows!
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![Diagram of wall displacement](image)

**Observation:**
- The change in the tube’s cross-sectional area is a second-order effect:
  \[ A = A_{undef} + \mathcal{O}(\epsilon^2). \]
- For small wall displacements, the oscillatory wall motion does not drive any (net) axial flows!
  
- [Formally:
  \[ \alpha^{1/2} \ll \frac{L}{a} \ll \frac{1}{\alpha \epsilon}, \]
  allows us to neglect axial flows and to replace \( \nabla^2 \) by its 2D equivalent.]
3D to 2D: “Executive summary”

In the parameter regime considered here:

- Unsteady flow uncouples from mean flow.
- Unsteady flow is dominated by its 2D transverse components
- No coupling between the 2D transverse flows in the tube’s cross-sections.

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\text{(unsteady)} = \text{(steady)} + \text{(unsteady)}
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**3D to 2D: “Executive summary”**

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- No coupling between the 2D transverse flows in the tube’s cross-sections.

\[
\text{(unsteady)} = \text{(steady)} + \text{(unsteady)} \implies 2D \text{ problem in each transverse cross section.}
\]
Results: Numerics and asymptotics

- Use Heil & Hazel’s `oomph-lib` library to solve the ALE formulation of the 2D Navier-Stokes equations.
  - Equations discretised with LBB-stable, quadrilateral Q2Q-1 elements.
  - Time-stepping performed with BDF4 scheme (BDF2 for FSI cases).
  - Unstructured quadtree mesh refinement, based on Z2 error estimation, to resolve the thin Stokes layers that develop near the wall.

- Asymptotics solution of the unsteady Stokes equations, based on expansions in inverse powers of $\alpha$ (SLW).
**Velocity field: Numerics vs asymptotics**

Velocity and pressure fields for $\alpha^2 = 100, \epsilon = 0.1, N = 2, \Lambda = -0.5$

<table>
<thead>
<tr>
<th>FE simulation</th>
<th>Composite asymptotic solution</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="FE simulation" /></td>
<td><img src="image2" alt="Composite asymptotic solution" /></td>
</tr>
<tr>
<td><img src="image3" alt="FE simulation" /></td>
<td><img src="image4" alt="Composite asymptotic solution" /></td>
</tr>
</tbody>
</table>
FSI simulations

Procedure:

- Initial conditions:
  - Wall deformed statically into buckled shape with amplitude $\varepsilon$.
  - Fluid is at rest.
- At $t = 0$ ‘release’ the wall and follow the system’s evolution.
- Characterise the system’s state by plotting the time evolution of the control radius $R_{ctrl}$.
Results

Evolution of control radius for $\alpha^2 = 100$:

- Damped harmonic oscillation.
- Period of oscillation $= \mathcal{O}(1)$ $\Rightarrow$ scaling estimates were correct.
- $\Rightarrow$ Fluid inertia is indeed balanced by wall stiffness.
- Slightly dull...
Results

Evolution of control radius for $\alpha^2 = 50$:

- $\alpha \propto \mu^{-1/2}$.
- Smaller $\alpha \iff$ Larger dissipation. $\iff$ More rapid decay of the oscillation.

What happens at $t \approx 11$ ??
Results

Evolution of control radius for $\alpha^2 = 10$:

- $\alpha \propto \mu^{-1/2}$.
- Smaller $\alpha \iff$ Larger dissipation. $\iff$ More rapid decay of the oscillation.

What happens at $t \approx 3.5$  

???
What is going on?

Consider case with $\alpha^2 = 10$:

```
\begin{tikzpicture}
  \begin{axis}[
    xlabel={time},
    ylabel={control radius $R_{\text{ctrl}}$},
    xmin=0, xmax=6,
    ymin=0.9, ymax=1.1,
    xtick={1,2,3,4,5,6},
    ytick={0.9,0.95,1,1.05,1.1},
    \]
  \addplot[red, thick] coordinates {
    (0,1.0)
    (1,1.1)
    (2,1.05)
    (3,0.95)
    (4,1.0)
    (5,1.1)
    (6,1.05)
  };
  \end{axis}
\end{tikzpicture}
```

“Type I” oscillation:

Initially:
What is going on?

Consider case with $\alpha^2 = 10$:

"Type I" oscillation:
What is going on?

“Type I” oscillation:

Initial configuration:

- Cross-sectional area $A < A_{undef}$.
- Change in cross-sectional area is second-order effect but Navier-Stokes equations conserve volume (area) exactly [...and so does our code!].
What is really going on?

Consider case with $\alpha^2 = 10$:

```
<table>
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<tr>
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</tr>
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<tr>
<td>1</td>
<td>0.9</td>
</tr>
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</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1.05</td>
</tr>
<tr>
<td>5</td>
<td>1.1</td>
</tr>
</tbody>
</table>

“Type I” oscillation at constant volume (area):

• When passing through the axisymmetric state, the wall is strongly compressed.

• $\Longrightarrow$ Local maximum in strain energy $\Pi_{\text{strain}}$. 
This is going on:

Evolution of control displacement and energy budget for $\alpha^2 = 10$:

- Note maxima in strain energy $\Pi_{\text{strain}}$ whenever the wall is in its (approximately) axisymmetric and its most strongly buckled configurations.
This is going on:

Evolution of control displacement and energy budget for $\alpha^2 = 10$:

- Initially: Large total energy $\Pi_{total} \gg \Pi^{(\text{barrier})}_{\text{strain}}$ allows the system to overcome the potential energy barrier associated with the approximately axisymmetric state.
This is going on:

Evolution of control displacement and energy budget for $\alpha^2 = 10$:

- Viscous dissipation causes (rapid) decay of the total energy:

$$\frac{d\Pi_{\text{total}}}{dt} = -D.$$
This is going on:

Evolution of control displacement and energy budget for $\alpha^2 = 10$:

- Once $\Pi_{\text{total}} < \Pi_{\text{strain}}^{(\text{barrier})}$, system becomes ‘trapped’ in one of the two buckled states.

"Type II" oscillation at constant volume (area):
Oscillations about axisymmetric equilibrium states

- Transition from type I to type II oscillation can only be avoided if the initial configuration satisfies $A = A_{undef}$.

- Start simulation from undeformed state and initiate oscillations with transient perturbation

$$f_{\text{transient}} = \begin{cases} 
  p \cos \cos(Nx^2) \mathbf{N} & \text{for } t \leq 0.3 \\
  0 & \text{for } t > 0.3
\end{cases}$$

Evolution for $\alpha^2 = 200$: Beautiful exponential decay to axisymmetric state.
**Analytical predictions for decay rates**

- Multiple-scales analysis for decay rate (SLW): Amplitude decays like

\[ A \sim \exp(kt) \]

- Compare against numerical results:
Conclusions

- A dynamic balance between fluid inertia and wall stiffness can support high-frequency, small-amplitude oscillations so that:

\[
\text{(unsteady)} + \text{(steady)} = \text{(unsteady)}
\]

- In the 2D system studied here, oscillations inevitably decay but they do so in an ‘interesting’ and generic manner.
- The transition between type I and type II oscillations is not an artefact of the 2D system!
- Decay rates indicate how much energy we have to extract from the flow (by other mechanisms) to maintain the oscillations.
- Parameter values required for the oscillations analysed here can be realised experimentally for sufficiently stiff tubes.
The two types of oscillation are an artefact of the 2D system...(?)

Admittedly...

- ...in 3D, the area constraint does not exist because axial inflow into the cross-sections is possible.

But...
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- ...the scenario observed here is generic for oscillations of externally compressed collapsible tubes.
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Recall:
- Transition from type I to type II oscillation is due to a local maximum in the strain energy of the axisymmetric state.
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Recall:
- Transition from type I to type II oscillation is due to a local maximum in the strain energy of the axisymmetric state.
- In the 3D case, the maximum in the strain energy of the axisymmetric state can be inferred from its static instability.
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  Example: Decaying large-amplitude oscillation.

- System approaches (buckled!) equilibrium state.
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Example: Decaying large-amplitude oscillation.

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Example: Large-amplitude oscillation develops from linearly unstable buckled equilibrium state.

- Initial stages of the oscillation must be of type I:
**The two types of oscillation are an artefact of the 2D system...**

Example: Large-amplitude oscillation develops from linearly unstable buckled equilibrium state.

- Initial stages of the oscillation must be of type I:
OK – but what about the parameter values?

- Are sufficiently large values of Strouhal and Womersley numbers ever observed in ‘real’ collapsible tubes?
OK – but what about the parameter values?

Parameter estimates:

- Fluid density (water):
  \[ \rho = 1000 \, \text{kg/m}^3 \]

- Fluid viscosity (water):
  \[ \mu = 1.0 \times 10^{-3} \, \text{kg/(m sec)} \]

- Elastic modulus and Poisson ratio (rubber)
  \[ E = 1.1 \times 10^6 \, \text{Pa} \quad \text{and} \quad \nu = 0.49 \]
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Relatively thin-walled tube [Heil, JFM 353]

- Tube radius
  \[ a = 4.2 \times 10^{-3} \text{ m} \]

- Wall-thickness-to-radius ratio
  \[ h/a = 0.1 \]
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\[ \alpha = \left( \frac{a}{\mu} \right)^{1/2} \left( K \rho \right)^{1/4} = 38.20 \]
OK – but what about the parameter values?

Relatively thick-walled tube [Bertram et al., JFS 4]

- Tube radius
  \[ a = 6.5 \times 10^{-3} m \]

- Wall-thickness-to-radius ratio
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Relatively thick-walled tube [Bertram et al., JFS 4]

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- Wall-thickness-to-radius ratio

\[ \frac{h}{a} = 0.3 \]

\[
\alpha = \left( \frac{a}{\mu} \right)^{1/2} (K \rho)^{1/4} = 108.3
\]
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- Tube radius
  \[ a = 6.5 \times 10^{-3} \text{m} \]

- Wall-thickness-to-radius ratio
  \[ h/a = 0.3 \]

- Flow rate
  \[ Q = 180 \times 10^{-3} \text{Liter/sec} \]

- Similar values directly from \( St = a/(UT) \) and measured periods of oscillation.
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  \[ Q = 180 \times 10^{-3} \text{Liter/sec} \]

\[
St = \frac{1}{U} \left( \frac{K}{\rho} \right)^{1/2} = 1.33
\]

- Similar values directly from \( St = a/(UT) \) and measured periods of oscillation.

- Not huge, but Jensen & Heil found that large Strouhal-number theory works perfectly for \( St = 0.5 \) and still catches the essential physics (macroscopically, little interaction between mean flow and oscillation) at \( St = 0.05 \).

- [ This doesn’t prove anything but it’s encouraging...]