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# The energetics of flow through a rapidly oscillating tube with slowly varying amplitude

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Motivated by the problem of self-excited oscillations in fluid-filled collapsible tubes, we examine the flow structure and energy budget of flow through an elastic-walled tube. Specifically, we consider the case in which a background axial flow is perturbed by prescribed small-amplitude high-frequency long-wavelength oscillations of the tube wall, with a slowly growing or decaying amplitude. We use a multiple-scale analysis to show that, at leading order, we recover the constant-amplitude equations derived by Whittaker *et al.* (Whittaker *et al.* 2010 *J. Fluid Mech.* **648**, 83–121. (doi:10.1017/S0022112009992904)) with the effects of growth or decay entering only at first order. We also quantify the effects on the flow structure and energy budget. Finally, we discuss how our results are needed to understand and predict an instability that can lead to self-excited oscillations in collapsible-tube systems.

**Keywords:** fluid–structure interaction; Starling resistor; multiple-scale analysis

## 1. Introduction

There is much interest in the study of flow-induced instabilities in flow through elastic-walled tubes, as such systems occur in a wide range of industrial and biological applications. Recent reviews by Heil & Jensen [1] and Grotberg & Jensen [2] include the examples of pipe flutter, wheezing during forced expiration from the lungs and the development of Korotkoff sounds during blood pressure measurement.

Experimentally and theoretically, the canonical set-up for studying such flows is a device known as a Starling resistor. The device, depicted in figure 1, comprises a length of elastic tube fixed between two rigid pipes and placed inside a pressure chamber. Flow is driven through the tubing either by an applied pressure drop between the ends or by the use of a volumetric pump. The external pressure in the chamber can also be adjusted to control the degree of collapse of the elastic section.

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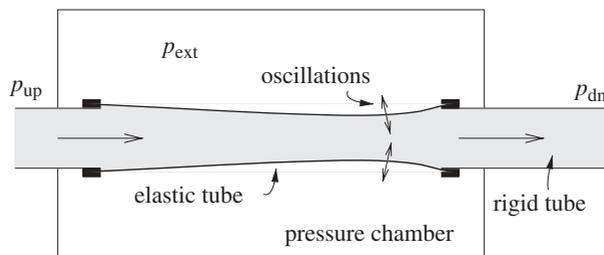


Figure 1. A sketch of a typical Starling resistor set-up. A length of flexible tubing is clamped between two rigid extensions and placed inside a pressure chamber. The external pressure and the upstream and downstream conditions can be adjusted to provide different flow regimes.

Experiments in such a system have long been known to exhibit large-amplitude self-excited oscillations in certain parameter regimes (see [3]). Numerical simulations of the two-dimensional equivalent of the Starling resistor have revealed a rich variety of behaviour [4,5], and more recently Heil & Boyle [6] have simulated fully time-dependent flow in the three-dimensional system. Lower dimensional models have also predicted self-excited oscillations through several different mechanisms [7–11].

Here, we consider a mechanism for self-excited oscillations that was first identified by Jensen & Heil [12], who studied high-frequency oscillations in a two-dimensional channel in which one section of the rigid wall is replaced by an elastic membrane. Briefly, this mechanism involves the growth of an oscillatory instability by the extraction of kinetic energy from the mean flow. Oscillatory motion of the elastic tube wall leads to internal volume changes, and in turn to an oscillatory perturbation to the mean axial velocity. An oscillatory flow at the far upstream and downstream ends of the system increases the kinetic energy flux at those ends in proportion to the square of the amplitude there. Therefore, if the amplitude of the oscillatory perturbation to the flow at the upstream end is larger than that at the downstream end, then there is a net energy flux into the system. If this net flux is larger than the additional dissipation caused by the oscillatory flow, then there is energy available to drive an instability. The difference in the oscillatory amplitude at the upstream and downstream ends is controlled by the partitioning of the axial oscillatory flux between the two ends. A difference can be caused by having different lengths of the rigid tubes at each end (a longer tube means a greater inertial resistance, and hence a smaller oscillatory amplitude), or by applying a flux condition at one end (in which case all the oscillatory flow occurs at the other end).

Whittaker *et al.* [13] considered the energy budget for prescribed small-amplitude, high-frequency, long-wavelength wall oscillations in a three-dimensional Starling resistor, and showed that the same mechanism for energy extraction also operates in that system. The paper derived an energy budget for the system, calculating the time-averaged energy flux  $\mathcal{E}$  to or from the tube wall required to sustain the prescribed constant-amplitude oscillations. The idea is then to apply this energy budget to the case of self-excited oscillations, determining the growth or decay rate by equating the excess energy that would have been transferred to the wall, in the case of prescribed oscillations, with

the change in the amplitude-dependent kinetic and elastic energy of the fluid and wall. Hence, if  $\mathcal{E} > 0$  for prescribed oscillations of fixed amplitude, we would expect the corresponding free-oscillation case to be unstable to that particular mode of oscillation, and vice versa.

However, Whittaker *et al.* [13] considered only the case of oscillations of constant amplitude. When considering self-excited oscillations, the amplitude will grow or decay. Additional terms are then present in the equations (compared with the constant-amplitude prescribed-wall-motion system) to account for the varying amplitude, and it is not obvious whether the constant-amplitude solutions are still applicable. The results of Whittaker *et al.* [13] allow us to find the neutrally stable (constant amplitude) modes, but further analysis is necessary to evaluate the growth or decay rates of each non-neutral mode, and hence determine which is the most unstable.

In this paper, we consider prescribed wall oscillations of time-varying amplitude, and determine how the fluid flow and energy budget differ from the solutions for constant-amplitude oscillations. The results derived here are used by Whittaker *et al.* [14] in the manner described above to analyse the full stability problem for self-excited oscillations.

In §2, we describe the mathematical set-up and define the small parameters that set the high-frequency, small-amplitude, long-wavelength asymptotic regime in which we work. In §3, we describe how the solutions to the equations governing the fluid flow are altered by the slow growth or decay of the prescribed oscillations. The leading-order solutions are found to be the same as in the constant-amplitude case, with changes occurring only at first order and beyond. In §4, we examine the energy budget for the system. The oscillations can induce changes in the non-oscillatory flow (via the nonlinear Reynolds stress term), and the kinetic energy associated with these changes can be of the same order as the energy changes associated with the growth of the oscillations. However, we show that the kinetic energy associated with the oscillatory flow decouples from the kinetic energy of the non-oscillatory component. Therefore, the changes in the non-oscillatory flow do not interfere with the main energy budget of the oscillations. Finally, we present some discussion of our results and their applications in §5.

## 2. Mathematical set-up

We adopt the ‘Starling resistor’ set-up and notation described in Whittaker *et al.* [13,15], and shown here in figure 2. Briefly, we consider flow through a flexible-walled tube of length  $L$  and typical diameter  $2a$ , whose wall is subject to prescribed oscillations. The position of the wall at time  $t$  is described by a vector function  $\mathbf{r}(Y, Z, t)$ , making use of Lagrangian coordinates  $(Y, Z)$  that parametrize the surface of the tube wall. In its undeformed configuration (which is not necessarily axially uniform), the wall has position  $\mathbf{r}_0(Y, Z)$ , and the tube is aligned with Cartesian coordinates  $(x, y, z)$  with its axis in the  $z$ -direction. The prescribed wall oscillations about this configuration have a typical time scale  $T$  and time-dependent amplitude. We restrict ourselves to exponentially growing or decaying harmonic oscillations of the form

$$\mathbf{r}(Y, Z, t) = \mathbf{r}_0(Y, Z) + \mathbf{r}'(Y, Z, t), \quad (2.1a)$$

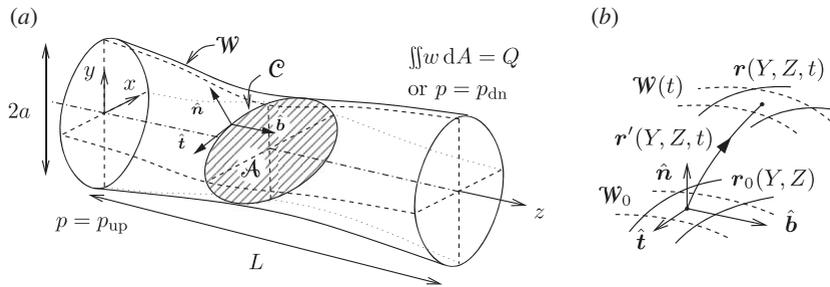


Figure 2. (a) A sketch of the flexible tube, showing the coordinates, dimensions and notation for the various surfaces and volumes. The centreline lies along  $x = y = 0$ , with  $z$  being the axial coordinate. The mean flow is from left to right, with a pressure boundary condition at the upstream end, and either a flux boundary condition or a second pressure condition at the downstream end. (b) A close-up of the upper surface of the tube, depicting the triad of unit vectors ( $\hat{n}$ ,  $\hat{t}$ ,  $\hat{b}$ ) aligned with the surface  $\mathcal{W}_0$  in the steady configuration, and the displacement vector  $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$ .

where

$$\mathbf{r}'(Y, Z, t) = d\tilde{\mathbf{r}}(Y, Z) e^{\sigma t} e^{i\omega t}. \quad (2.1b)$$

The growth rate  $\sigma$  and frequency  $\omega$  are real constants, and real parts are assumed. The  $O(1)$  (possibly) complex-valued function  $\tilde{\mathbf{r}}(Y, Z)$  represents the mode shape. We keep the frequency  $\omega$  and time scale  $T$  as two independent variables to aid comparison with the case of self-excited oscillations (where the time scale, but not the precise frequency, can be predicted in advance using scaling arguments). We envisage a situation of slow growth, so that  $\sigma \ll \omega$ .

The set of points occupied by the tube wall at time  $t$  is denoted  $\mathcal{W}(t)$ . The (time dependent) cross section at each axial position is denoted  $\mathcal{A}(z, t)$ , and its area  $A(z, t)$ . The perimeter of the tube wall bounding the cross section at  $z$  is  $\mathcal{C}(z, t)$ , and its length is  $C(z, t)$ . The region occupied by the fluid in the tube is  $\mathcal{V}(t)$ . We define corresponding quantities  $\mathcal{W}_0$ ,  $\mathcal{A}_0(z)$ ,  $A_0(z)$ ,  $\mathcal{C}_0(z)$ ,  $C_0(z)$  and  $\mathcal{V}_0$  to be the corresponding values in the undeformed configuration in the absence of any oscillations.

The tube is filled with fluid of density  $\rho$  and dynamic viscosity  $\mu$ . A mean axial flow, with typical velocity  $\mathcal{U}$ , is driven along the tube either by a prescribed flux  $Q$  at the downstream end or by a prescribed pressure difference  $p_{\text{up}} - p_{\text{dn}}$  between the two ends. The value of  $\mathcal{U}$  is chosen so that, in the absence of oscillations, the steady volume flux is  $\pi a^2 \mathcal{U}$ . (So for the flux case  $\mathcal{U} = Q/(\pi a^2)$ , whereas for the case of an imposed pressure drop,  $\mathcal{U}$  must be determined from  $p_{\text{up}} - p_{\text{dn}}$  by solving a steady flow problem.)

The fluid velocity  $\mathbf{u}$  and pressure  $p$  inside the tube are governed by the Navier–Stokes equations

$$\nabla \cdot \mathbf{u} = 0 \quad (2.2a)$$

and

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u}. \quad (2.2b)$$

Boundary conditions arise from the kinematic condition on the tube wall

$$\mathbf{u} = \frac{\partial \mathbf{r}}{\partial t} \quad \text{on} \quad \mathcal{W}, \quad (2.3)$$

and the pressure or flux conditions at the tube ends,

$$p = p_{\text{up}} \quad \text{at} \quad z = 0, \quad (2.4a)$$

and either

$$p = p_{\text{dn}} \quad \text{or} \quad \iint_{\mathcal{A}} \mathbf{u} \cdot \hat{\mathbf{z}} dA = Q \quad \text{at} \quad z = L. \quad (2.4b,c)$$

(These tube-end boundary conditions formally leave the system under-determined, but this will turn out not to matter when we later move to a long-wavelength limit.)

(a) *The growth rate for the prescribed oscillations*

We would like to set the frequency  $\omega$  and the growth rate  $\sigma$  so that they are comparable with those expected for the case of self-excited oscillations in the same system. The frequency is determined by a balance between fluid inertia and the restoring force from the walls [14], and so is undetermined here (as the wall mechanics are not specified). This turns out not to matter, as we can scale all times on the time scale  $T$  for the oscillation period, and absorb this scale into the other dimensionless parameters in the system. Having chosen this time scale, we do, however, need to choose an appropriate scale for the growth or decay rate of the oscillations. We estimate this scale from energy considerations as follows.

Whittaker *et al.* [13] showed that the dimensional scale for the oscillatory kinetic energy flux  $F$  at the upstream end of the tube is formed by the product of the fluid density, the mean axial velocity, the square of the oscillatory velocity amplitude and the cross-sectional area. Specifically, it was shown that

$$F \sim \rho \mathcal{U} \left( \frac{a \Delta L a}{a^2 T} \right)^2 a^2 = \frac{\rho \mathcal{U} d^2 L^2}{T^2}. \quad (2.5)$$

The kinetic energy  $E_f$  of the oscillatory flow is estimated as the product of the fluid density, the square of the oscillatory velocity amplitude and the volume of the tube. This leads to

$$E_f \sim \rho \left( \frac{a \Delta L a}{a^2 T} \right)^2 a^2 L = \frac{\rho d^2 L^3}{T^2}. \quad (2.6)$$

We can now use equations (2.5) and (2.6) to estimate the time scale for growth under the assumption that the rate of increase of oscillatory kinetic energy is comparable with the flux of kinetic energy at the upstream end. Balancing the rate of change of  $E_f$  with the flux  $F$ , we obtain

$$\sigma \sim \frac{F}{E_f} \sim \frac{\mathcal{U}}{L}. \quad (2.7)$$

The fact that this scale is independent of the amplitude  $d$  justifies the assumption of exponential growth or decay, made in equations (2.1a,b).

*(b) Dimensionless groups and parameter regime*

The five key dimensionless groups in the problem are

$$\ell = \frac{L}{a}, \quad (2.8a)$$

$$\Delta = \frac{d}{a}, \quad (2.8b)$$

$$\alpha^2 = \frac{\rho a^2}{\mu T}, \quad (2.8c)$$

$$St = \frac{a}{\mathcal{U}T} \quad (2.8d)$$

and 
$$\Sigma = \sigma T. \quad (2.8e)$$

The two geometric ratios,  $\ell$  and  $\Delta$ , correspond to the tube length and oscillation amplitude, respectively. The next two groups relate to the fluid mechanics. The Womersley number  $\alpha$  gives the relative importance of unsteady inertia to viscous effects. The Strouhal number  $St$  indicates the relative importance of unsteady to convective inertia. Finally,  $\Sigma$  is the ratio of the oscillation time scale to the growth time scale.

We work in the parameter regime in which the tube wall performs small-amplitude, high-frequency, long-wavelength oscillations. To formalize this regime, we now introduce small parameters  $\epsilon$  and  $\delta$ , and  $O(1)$  quantities  $\lambda$ ,  $\beta$  and  $R$ . We then represent the first four dimensionless groups as

$$\ell^2 = \frac{1}{\epsilon\beta} \gg 1, \quad (2.9a)$$

$$\Delta = \epsilon\delta \ll 1, \quad (2.9b)$$

$$\alpha = \frac{R}{\epsilon} \gg 1 \quad (2.9c)$$

and 
$$\ell St = \frac{\lambda}{\epsilon} \gg 1. \quad (2.9d)$$

These are the same scalings as were used by Whittaker *et al.* [13]. The sizes of  $\alpha$  and  $St$  are chosen to ensure that the instability mechanism described in the introduction can operate efficiently. The sizes of  $\ell$  and  $\delta$  are chosen for mathematical convenience.

Assuming that the growth rate  $\sigma$  follows the estimate (2.7), we have  $\Sigma \sim \mathcal{U}T/L = (\ell St)^{-1}$ . We therefore introduce a final  $O(1)$  parameter  $\mathcal{A}$ , and write

$$\Sigma = \frac{\epsilon\mathcal{A}}{\lambda} \ll 1. \quad (2.10)$$

With this scaling, the growth rate is much smaller than the frequency of the oscillations. This allows us to use a multiple-scale approach when analysing the system.

(c) *Non-dimensionalization and flow-field decomposition*

We non-dimensionalize all axial lengths on the tube length  $L$  and all transverse lengths on the radial scale  $a$ , writing

$$(x, y, z) = (a\check{x}, a\check{y}, L\check{z}), \quad (2.11a)$$

$$A_0 = a^2 \check{A}_0, \quad (2.11b)$$

and 
$$A = a^2 \check{A}. \quad (2.11c)$$

Times are non-dimensionalized using the oscillation time scale  $T$ , so we write

$$t = T \quad (2.11d)$$

and

$$\omega = T^{-1} \check{\omega}. \quad (2.11e)$$

We also introduce a slow time variable  $\check{\tau}$  to capture the growth or decay of the oscillation, as this takes place over a much longer time scale. Motivated by equation (2.10), we define

$$\check{\tau} = \frac{\epsilon}{\lambda} \check{t}. \quad (2.12)$$

Time derivatives can then be expressed as

$$\frac{\partial}{\partial t} = \frac{1}{T} \frac{\partial}{\partial \check{t}} = \frac{1}{T} \left( \frac{\partial}{\partial \check{t}} \Big|_{\check{\tau}} + \frac{\epsilon}{\lambda} \frac{\partial}{\partial \check{\tau}} \Big|_{\check{t}} \right). \quad (2.13)$$

Following Whittaker *et al.* [13], we write the fluid velocity as  $\mathbf{u} = \mathbf{u}^\perp + w\hat{\mathbf{z}}$ , and non-dimensionalize the axial component  $w$ , the transverse component  $\mathbf{u}^\perp$  and pressure  $p$  by writing

$$w = \mathcal{U} \check{w}, \quad (2.14a)$$

$$\mathbf{u}^\perp = \frac{\mathcal{U}}{\ell} \check{\mathbf{u}}^\perp \quad (2.14b)$$

and 
$$p - p_{\text{up}} = \frac{\rho L \mathcal{U}}{T} \check{p}. \quad (2.14c)$$

The oscillating tube wall results in oscillations in the cross-sectional area. We introduce a non-dimensional amplitude  $\check{A}(\check{z})$  for these oscillations, and write

$$\check{A}(\check{z}, \check{t}, \check{\tau}) = \check{A}_0(\check{z}) + \epsilon \delta \check{A}(\check{z}) e^{A\check{\tau}} e^{i\check{\omega}\check{t}} + \dots \quad (2.15)$$

We also decompose the other variables into steady and oscillatory components, and expand in powers of  $\epsilon$  and  $\delta e^{A\tilde{\tau}}$  as follows:

$$\begin{aligned} \check{w} = & [\bar{w}_0 + (\delta e^{A\tilde{\tau}})^2 \bar{w}_2 + \dots] \\ & + \lambda \delta e^{A\tilde{\tau}} [\tilde{w}_{00} + \epsilon \tilde{w}_{01} + \dots] e^{i\check{\omega}\tilde{t}} + \lambda \epsilon (\delta e^{A\tilde{\tau}})^2 \hat{w}, \end{aligned} \quad (2.16a)$$

$$\begin{aligned} \check{\mathbf{u}}^\perp = & [\bar{\mathbf{u}}_0^\perp + (\delta e^{A\tilde{\tau}})^2 \bar{\mathbf{u}}_2^\perp + \dots] \\ & + \lambda \delta e^{A\tilde{\tau}} [\tilde{\mathbf{u}}_{00}^\perp + \epsilon \tilde{\mathbf{u}}_{01}^\perp + \dots] e^{i\check{\omega}\tilde{t}} + \lambda \epsilon (\delta e^{A\tilde{\tau}})^2 \hat{\mathbf{u}}^\perp \end{aligned} \quad (2.16b)$$

and

$$\begin{aligned} \check{p} = & \frac{\epsilon}{\lambda} [\bar{p}_0 + (\delta e^{A\tilde{\tau}})^2 \bar{p}_2 + \dots] \\ & + \lambda \delta e^{A\tilde{\tau}} [\tilde{p}_{00} + \epsilon \tilde{p}_{01} + \dots] e^{i\check{\omega}\tilde{t}} + \lambda \epsilon (\delta e^{A\tilde{\tau}})^2 \hat{p}. \end{aligned} \quad (2.16c)$$

Here, overbars denote non-oscillatory components, tildes the oscillatory perturbations of non-dimensional frequency  $\check{\omega}$ , and hats the oscillatory components of higher frequency. We scale the velocity components using the natural scales from the mean flow and wall motion. For the pressure, we use appropriate inertial scales for the steady and oscillatory components.

Following Whittaker *et al.* [13], it is convenient to expand the non-oscillatory terms ( $\bar{w}$ ,  $\bar{\mathbf{u}}^\perp$ ,  $\bar{p}$ ) in equations (2.16a–c) just in powers of  $\delta e^{A\tilde{\tau}}$ . Only even powers are required owing to the nature of the forcing through the nonlinear inertia terms. For the primary oscillatory terms ( $\tilde{w}$ ,  $\tilde{\mathbf{u}}^\perp$ ,  $\tilde{p}$ ), a double expansion in  $\epsilon$  and  $\delta e^{A\tilde{\tau}}$  is used, but we only require the first corrections, which turn out to involve only terms of  $O(\epsilon)$ . (The  $O(\delta)$  terms  $\tilde{w}_{10}$ ,  $\tilde{\mathbf{u}}_{10}^\perp$  and  $\tilde{p}_{10}$  all vanish because of the nature of the coupling.) We do not consider the higher frequency oscillatory terms ( $\hat{w}$ ,  $\hat{\mathbf{u}}^\perp$ ,  $\hat{p}$ ) in detail, so no expansion is necessary for them.

Finally, energy and energy fluxes are non-dimensionalized on

$$\rho \mathcal{U}^2 a^2 L \quad (2.17a)$$

and

$$\rho \mathcal{U}^3 a^3, \quad (2.17b)$$

based on the kinetic energy and kinetic energy fluxes in the steady flow.

### 3. The fluid equations

We substitute the expansions (2.16a–c) into the Navier–Stokes equations (2.2a,b) and collect terms with the same temporal frequency in the fast time variable  $\tilde{t}$ , expanding products of Fourier terms that arise through the nonlinear terms. Each frequency component is then solved as an asymptotic expansion in the small parameters  $\epsilon$  and  $\delta$ .

(a) *The non-oscillatory component at leading order*

The non-oscillatory component of the Navier–Stokes equations comprises the terms that are steady in the fast time variable  $\check{t}$ . At leading order in  $\delta$ , we have

$$\check{\nabla}_{\perp} \cdot \bar{\mathbf{u}}_0^{\perp} + \frac{\partial \bar{w}_0}{\partial \check{z}} = 0, \quad (3.1a)$$

$$\frac{\partial \bar{\mathbf{u}}_0}{\partial \check{\tau}} + (\bar{\mathbf{u}}_0 \cdot \check{\nabla}) \bar{\mathbf{u}}_0^{\perp} = -\frac{1}{\epsilon\beta} \check{\nabla}_{\perp} \bar{p}_0 + \frac{\lambda\epsilon}{R^2} \check{\nabla}_{\perp}^2 \bar{\mathbf{u}}_0^{\perp} + \frac{\lambda\beta\epsilon^2}{R^2} \frac{\partial^2 \bar{\mathbf{u}}_0^{\perp}}{\partial \check{z}^2} \quad (3.1b)$$

and

$$\frac{\partial \bar{w}_0}{\partial \check{\tau}} + (\bar{\mathbf{u}}_0 \cdot \check{\nabla}) \bar{w}_0 = -\frac{\partial \bar{p}_0}{\partial \check{z}} + \frac{\lambda\epsilon}{R^2} \check{\nabla}_{\perp}^2 \bar{w}_0 + \frac{\lambda\beta\epsilon^2}{R^2} \frac{\partial^2 \bar{w}_0}{\partial \check{z}^2}, \quad (3.1c)$$

where  $\bar{\mathbf{u}}_0 \equiv \bar{\mathbf{u}}_0^{\perp} + \ell \bar{w}_0 \hat{\mathbf{z}}$  is the full leading-order non-oscillatory velocity. The boundary conditions are

$$\bar{\mathbf{u}}_0 = \mathbf{0} \quad \text{on} \quad \mathcal{W}_0 \quad (3.2a)$$

and

$$\bar{p}_0 = 0 \quad \text{at} \quad \check{z} = 0, \quad (3.2b)$$

and either

$$\iint_{\mathcal{A}_0} \bar{w}_0 \, dA = \pi \quad \text{or} \quad \bar{p}_0 = -\frac{\epsilon\lambda}{R^2} \frac{a^2(p_{\text{up}} - p_{\text{dn}})}{\mu\mathcal{U}L} \quad \text{at} \quad \check{z} = 1, \quad (3.2c,d)$$

for a flux or pressure condition at the downstream end. The system (3.1a–c) is a non-dimensional statement of the full Navier–Stokes equations for  $\bar{\mathbf{u}}$ , but with only the slow time derivative present.

There is no explicit dependence on the slow time variable  $\check{\tau}$  in equations (3.1a–c) and (3.2a–d), and hence we expect a purely steady solution. We have the same set of equations and boundary conditions as considered in Whittaker *et al.* [13], and, at leading order, the steady flow is not affected by the oscillations or their growth or decay. It turns out that the precise details of the non-oscillatory flow are not important for the oscillatory-flow and energy-budget calculations we shall perform below. The only information about the non-oscillatory flow that matters is the dimensionless axial volume flux, which is

$$\mathbb{Q}_0 \equiv \iint_{\mathcal{A}_0} \bar{w}_0 \, dA = \pi, \quad (3.3)$$

by virtue of our choice of velocity scale  $\mathcal{U}$ .

The  $O(\delta^2)$  corrections, however, necessarily involve variations on the slow time scale, as a factor of  $e^{A\check{\tau}}$  appears with every  $\delta$ . The equations are forced at this order by the contributions that the leading-order oscillatory flow makes via the nonlinear inertia terms (Reynolds stresses). Therefore, before we can analyse the  $O(\delta^2)$  corrections, we must first determine the oscillatory flow.

(b) *The fundamental oscillatory component at leading order*

The fundamental oscillatory component contains the terms in the Navier–Stokes equations that are proportional to  $e^{i\omega\check{t}}$  on the fast time scale. Here we consider those terms that appear at leading order in both  $\epsilon$  and  $\delta$ .

The time derivative of any term in the oscillatory component involves both the fast and slow time scales by virtue of the factor of  $e^{A\tilde{\tau}}e^{i\omega\tilde{t}}$  multiplying those terms in equations (2.16*a–c*). However, as can be seen from equation (2.13), the derivatives with respect to the slow time  $\tilde{\tau}$  appear with an additional factor of  $\epsilon$ , so they do not contribute at leading order.

As in Whittaker *et al.* [13], it is convenient to split each of the oscillatory pressures  $\tilde{p}_{0i}$  into a cross-sectionally averaged component  $\tilde{p}_{0i}^*$ , and the component  $\tilde{p}_{0i}^\Delta$  containing the cross-sectional variation. We write

$$\tilde{p}_{0i} = \tilde{p}_{0i}^*(z) + \tilde{p}_{0i}^\Delta, \quad \text{where} \quad \tilde{p}_{0i}^*(z) = \frac{1}{A_0(z)} \iint_{\mathcal{A}_0} \tilde{p}_{0i}^* \, dA. \quad (3.4)$$

The leading-order equations for the oscillatory flow are the same as those derived in Whittaker *et al.* [13], and we therefore obtain the same solutions for  $\tilde{u}_{00}$ ,  $\tilde{\mathbf{u}}_{00}^\perp$ ,  $\tilde{p}_{00}$  and  $\tilde{p}_{01}^\Delta$ .<sup>1</sup> We find that there is an inviscid flow in the core, with a thin viscous Stokes layer adjacent to the tube wall. In the core,  $\tilde{u}_{00}$  and  $\tilde{p}_{00}$  are only functions of the axial coordinate  $\tilde{z}$ , and are governed by

$$\frac{d}{d\tilde{z}}(\tilde{u}_{00}A_0) = -i\omega\tilde{A} \quad (3.5a)$$

and

$$\frac{d\tilde{p}_{00}}{d\tilde{z}} = -i\omega\tilde{u}_{00}, \quad (3.5b)$$

representing conservation of mass and axial momentum, respectively. The boundary conditions on equations (3.5*a,b*) at the tube ends are either  $\tilde{p}_{00} = 0$  for a pressure condition, or  $\tilde{u}_{00} = 0$  for a flux condition. Since  $\tilde{A}(\tilde{z})$  is known from the prescribed forcing, equations (3.5*a,b*) can now be solved explicitly. The viscous Stokes layer is found to behave purely passively at this order, smoothing the axial plug flow in the core to zero on the wall to satisfy the no-slip boundary condition.

The transverse flow in the cross section is given by

$$\tilde{\mathbf{u}}_{00}^\perp = \frac{i}{\omega\beta} \nabla_\perp \tilde{p}_{01}^\Delta, \quad (3.6)$$

with the cross-sectional pressure variation  $\tilde{p}_{01}^\Delta$  found as the solution of a two-dimensional Poisson problem, which can be solved independently in each cross section. The details can be found in Whittaker *et al.* [13], but since the results are not important for the present work, we shall not repeat them here.

<sup>1</sup> $\tilde{p}_{01}^*(z)$  is determined at the next order in  $\epsilon$ ; see §3*c*.

(c) *The fundamental oscillatory component at first order*

At  $O(\epsilon)$ , terms from the slow time derivatives enter, in addition to the other correction terms seen in Whittaker *et al.* [13]. The governing equations are

$$\check{\nabla}_{\perp} \cdot \tilde{\mathbf{u}}_{01}^{\perp} + \frac{\partial \tilde{u}_{01}}{\partial \check{z}} = 0, \quad (3.7a)$$

$$i\check{\omega}\beta\tilde{\mathbf{u}}_{01}^{\perp} + \frac{\beta}{\lambda}[A\tilde{\mathbf{u}}_{00}^{\perp} + (\tilde{\mathbf{u}}_0 \cdot \check{\nabla})\tilde{\mathbf{u}}_{00}^{\perp} + (\tilde{\mathbf{u}}_{00} \cdot \check{\nabla})\tilde{\mathbf{u}}_0^{\perp}] = -\check{\nabla}_{\perp}\tilde{p}_{02}^{\Delta} \quad (3.7b)$$

and 
$$i\check{\omega}\tilde{w}_{01} + \frac{1}{\lambda}[A\tilde{w}_{00} + (\tilde{\mathbf{u}}_0 \cdot \check{\nabla})\tilde{w}_{00} + (\tilde{\mathbf{u}}_{00} \cdot \check{\nabla})\tilde{w}_0] = -\frac{\partial \tilde{p}_{01}^{\Delta}}{\partial \check{z}} - \frac{d\tilde{p}_{01}^*}{d\check{z}}, \quad (3.7c)$$

where we recall that  $\tilde{\mathbf{u}}_0 = \tilde{\mathbf{u}}_0^{\perp} + \ell\tilde{w}_0\hat{\mathbf{z}}$  is the leading-order steady velocity that was introduced in §3a above. (The terms arising from the slow time derivatives in equations (3.7a–c) are those pre-multiplied by  $A$ .) The boundary condition on the tube wall comes from matching with the flow in the Stokes layer. We find that

$$\hat{\mathbf{n}} \cdot (\tilde{\mathbf{u}}_{01}^{\perp} + \ell\tilde{w}_{01}\hat{\mathbf{z}}) = \tilde{U}_{01}^{\infty} \quad \text{on } \mathcal{C}_0, \quad (3.8a)$$

where  $\hat{\mathbf{n}}$  is the normal to the undeformed tube wall and  $\tilde{U}_{01}^{\infty}$  is the limiting normal velocity at the edge of the Stokes layer.<sup>2</sup> The boundary conditions at the tube ends are

$$\tilde{p}_{01} = 0 \quad \text{at } \check{z} = 0, \quad (3.8b)$$

from the upstream pressure condition (2.4a), and either

$$\iint_{\mathcal{A}_0} \tilde{w}_{01} dA = 0 \quad \text{or} \quad \tilde{p}_{01} = 0 \quad \text{at } \check{z} = 1, \quad (3.8c,d)$$

from equations (2.4b,c) at the downstream end. Finally, the decomposition (3.4) requires

$$\iint_{\mathcal{A}_0} \tilde{p}_{02}^{\Delta} dA = 0. \quad (3.9)$$

This system of equations for  $\tilde{w}_{01}$ ,  $\tilde{\mathbf{u}}_{01}$ ,  $\tilde{p}_{01}^*$  and  $\tilde{p}_{02}^{\Delta}$  can be solved in the same way as described in Whittaker *et al.* [13]. Moreover, inspecting the leading-order equations (3.5b) and (3.6), we find that we can absorb the additional terms involving  $A$  into the pressures in equations (3.7b) and (3.7c). On solving the system, we find that the velocities  $\tilde{w}_{01}$  and  $\tilde{\mathbf{u}}_{01}$  are unaffected by the slow growth, whereas

$$\tilde{p}_{01}^* = \tilde{p}_{01}^*|_{A=0} - \frac{iA}{\lambda\omega}\tilde{p}_{00}^* \quad (3.10a)$$

and

$$\tilde{p}_{02}^{\Delta} = \tilde{p}_{02}^{\Delta}|_{A=0} - \frac{iA}{\lambda\omega}\tilde{p}_{01}^{\Delta}. \quad (3.10b)$$

<sup>2</sup> $\tilde{U}_{01}^{\infty}$  can be computed trivially using conservation of mass from the (known) leading-order oscillatory axial and azimuthal velocities in the Stokes layer; see §5.4 of Whittaker *et al.* [13].

(d) *The non-oscillatory component at  $O(\delta^2)$*

A complicating factor in the non-oscillatory component at  $O(\delta^2)$  is the contribution from the oscillatory flow via the nonlinear inertia term. As in Whittaker *et al.* [13], this term can be absorbed into a modified pressure. We write

$$\bar{p}' = \bar{p}_0 + (\delta e^{A\tilde{\tau}})^2 \bar{p}_2 + \frac{1}{4} \lambda^2 (\delta e^{A\tilde{\tau}})^2 (|\tilde{w}_{00}(\tilde{z})|^2 - |\tilde{w}_{00}(0)|^2) + O(\epsilon \delta^2, \delta^4) \quad (3.11a)$$

and

$$\bar{\mathbf{u}}' = \bar{\mathbf{u}}_0 + (\delta e^{A\tilde{\tau}})^2 \bar{\mathbf{u}}_2. \quad (3.11b)$$

The non-oscillatory components of the Navier–Stokes equations, correct to  $O(\delta^2)$ , can then be written as

$$\check{\nabla}_{\perp} \cdot \bar{\mathbf{u}}'^{\perp} + \frac{\partial \bar{w}'}{\partial \check{z}} = 0, \quad (3.12a)$$

$$\frac{\partial \bar{\mathbf{u}}'^{\perp}}{\partial \check{\tau}} + (\bar{\mathbf{u}}' \cdot \check{\nabla}) \bar{\mathbf{u}}'^{\perp} = -\frac{1}{\epsilon \beta} \check{\nabla}_{\perp} \bar{p}' + \frac{\lambda \epsilon}{R^2} \check{\nabla}_{\perp}^2 \bar{\mathbf{u}}'^{\perp} + \frac{\lambda \beta \epsilon^2}{R^2} \frac{\partial^2 \bar{\mathbf{u}}'^{\perp}}{\partial \check{z}^2} \quad (3.12b)$$

and

$$\frac{\partial \bar{w}'}{\partial \check{\tau}} + (\bar{\mathbf{u}}' \cdot \check{\nabla}) \bar{w}' = -\frac{\partial \bar{p}'}{\partial \check{z}} + \frac{\lambda \epsilon}{R^2} \check{\nabla}_{\perp}^2 \bar{w}' + \frac{\lambda \beta \epsilon^2}{R^2} \frac{\partial^2 \bar{w}'}{\partial \check{z}^2}, \quad (3.12c)$$

subject to the boundary conditions

$$\bar{p}' = 0 \quad \text{at} \quad \check{z} = 0, \quad (3.13a)$$

and either

$$\iint_{\mathcal{A}_0} \bar{w}' \, dA = \pi \quad \text{at} \quad \check{z} = 1, \quad (3.13b)$$

if we are imposing a flux condition at the downstream end, or

$$\begin{aligned} \bar{p}' = & -\frac{\epsilon \lambda}{R^2} \frac{a^2 (p_{\text{up}} - p_{\text{dn}})}{\mu \mathcal{U} L} \\ & - \frac{1}{4} \lambda^2 (\delta e^{A\tilde{\tau}})^2 (|\tilde{w}_{00}(0)|^2 - |\tilde{w}_{00}(1)|^2) + O(\epsilon \delta^2, \delta^4), \end{aligned} \quad (3.13c)$$

at  $\check{z} = 1$ , for the case of a second pressure condition there. The boundary condition on the tube wall is

$$\bar{\mathbf{u}}' = \mathbf{0} + O(\epsilon \delta^2, \delta^4) \quad \text{on} \quad \mathcal{W}_0. \quad (3.13d)$$

For the case of a downstream flux condition, the system of equations remains the same, so the steady solution at  $O(\delta^0)$  also satisfies the  $(\bar{\mathbf{u}}', \bar{p}')$  system at  $O(\delta^2)$ . The non-oscillatory pressure  $\bar{p}$  is modified according to equation (3.11a) to counteract the Reynolds stresses, but this occurs without affecting the fluid flow.

However, when we have a pressure condition at both ends of the tube, the pressure  $\bar{p}$  can no longer adjust to counteract the Reynolds stresses and still satisfy the tube-end boundary conditions. The conditions (3.13a) and (3.13c)

on the modified pressure  $\bar{p}'$  now yield a time-dependent (modified) pressure difference between the tube ends. This is equal to the  $O(\epsilon)$ -imposed steady pressure difference, plus a component proportional to  $\delta^2 e^{2A\tilde{\tau}}$ . We therefore have a more complicated time-dependent problem to solve.

Fortunately, it is unnecessary to solve this problem as scaling arguments can provide all the information we need. The flow evolution is dominated by a balance between the forcing pressure gradient, viscous drag and unsteady inertia in the axial momentum equation (3.12c). In a quasi-steady state, the pressure gradient  $\partial\bar{p}'/\partial\tilde{z}$  balances the viscous drag  $(\lambda\epsilon/R^2)\check{\mathbf{V}}_{\perp}^2\bar{w}'$ . If there is any mismatch, e.g. because of an increased pressure gradient, then the imbalance is taken up by the unsteady inertia term  $\partial\bar{w}'/\partial\tilde{\tau}$ . This causes the axial flow  $\bar{w}'$  to increase or decrease, until the viscous drag balances the pressure gradient in a new quasi-steady state.

We now consider how the axial flow will evolve as  $\tilde{\tau}$  increases from zero by an  $O(1)$  amount, corresponding to the growth time scale for the oscillations. During this time, the mean pressure gradient forced by the boundary condition (3.13c) changes by an  $O(\delta^2)$  amount. So there will be an unbalanced excess pressure gradient of at most  $O(\delta^2)$ , which means  $\partial\bar{w}'/\partial\tilde{\tau}$  can be at most  $O(\delta^2)$ . Hence, the axial velocity can change by at most  $O(\delta^2)$  in the  $O(1)$  slow time interval.<sup>3</sup>

While  $\delta e^{\lambda\tilde{\tau}}$  remains small, the actual adjustment to the flux also remains small. So over the  $\tilde{\tau} = O(1)$  time scale for appreciable growth in the amplitude to occur, there will be little change in the axial flux. Hence, when examining the onset of self-excited oscillations, the initial growth takes place as a perturbation to what is, at leading order, a steady axial flow with flux  $\mathbb{Q}_0 = \pi$ .<sup>4</sup>

#### 4. The energy budget for the fluid

The energy equation for the Navier–Stokes equations (2.2a,b) is

$$\begin{aligned} \frac{d}{dt} \left( \iiint_{\mathcal{V}} \frac{1}{2} \rho |\mathbf{u}|^2 dV \right) &= \left[ \iint_{\mathcal{A}} \mathbf{u} \cdot \left( -\frac{1}{2} \rho |\mathbf{u}|^2 \mathbf{I} - p \mathbf{I} + 2\mu \mathbf{e} \right) \cdot \hat{\mathbf{z}} dS \right]_{z=0}^{z=L} \\ &+ \iint_{\mathcal{W}} \mathbf{u} \cdot (-p \mathbf{I} + 2\mu \mathbf{e}) \cdot \hat{\mathbf{N}} dS \\ &- \iiint_{\mathcal{V}} 2\mu \mathbf{e} : \mathbf{e} dV, \end{aligned} \quad (4.1)$$

where  $\mathbf{e}$  is the rate-of-strain tensor and  $\mathbf{I}$  is the identity tensor. The left-hand side of equation (4.1) is the rate of change of kinetic energy in the fluid. The three integrals on the right-hand side represent the flux of kinetic energy and work done against the pressure and viscous forces at the tube ends, the work done at the tube walls, and the viscous dissipation, respectively.

<sup>3</sup>It can be seen that this is indeed the limiting effect, as a change of  $O(\delta^2)$  is much smaller than the  $O(\delta^2/\epsilon)$  change to  $\bar{w}'$  required to reach the quasi-steady state in equation (3.12c).

<sup>4</sup>The argument breaks down if  $A$  is too small, i.e. we are too close to the neutral stability point. If  $A = O(\delta^2)$  rather than  $O(1)$ , the growth time of  $\tilde{\tau} = O(A^{-1})$  is then long enough to allow for significant adjustment of the flux. See §5 for further details.

We use the non-dimensionalization introduced in §2*c*, and average equation (4.1) over a single period of the oscillation, to obtain

$$\begin{aligned}
 & \frac{d}{d\tilde{\tau}} \left[ \underbrace{\left[ \left[ \left[ \int_V \frac{1}{2} (\dot{w}^2 + \epsilon\beta |\dot{\mathbf{u}}^\perp|^2) dV \right] \right] \right]}_{\mathbb{E}} \Bigg]_0 \\
 &= \underbrace{\left[ \left[ \left[ \int_{\mathcal{A}} \frac{1}{2} (\dot{w}^2 + \epsilon\beta |\dot{\mathbf{u}}^\perp|^2) \dot{w} dA \right]_{\tilde{z}=1}^{\tilde{z}=0} \right] \right]}_{\mathcal{K}} \Bigg]_0 \\
 &- \underbrace{\left[ \left[ \left[ \int_{\mathcal{A}} \dot{w} \dot{p} - \frac{2\epsilon^2}{R^2} (\dot{\mathbf{u}} \cdot \dot{\mathbf{e}} \cdot \hat{\mathbf{z}}) dA \right]_{\tilde{z}=0}^{\tilde{z}=1} \right] \right]}_{\mathcal{J}} \Bigg]_0 \\
 &- \underbrace{\left[ \left[ \int_w \frac{\lambda}{\epsilon} \dot{p} \dot{\mathbf{u}} \cdot \hat{\mathbf{N}} - \frac{2\epsilon^2}{R^2} \left( \frac{1}{\ell} \dot{\mathbf{u}} \cdot \dot{\mathbf{e}} \cdot \hat{\mathbf{N}} \right) dS \right] \right]}_{\mathcal{E}} \Bigg]_0 - \underbrace{\left[ \left[ \int_V \frac{\epsilon\lambda}{R^2} 2\dot{\mathbf{e}} : \dot{\mathbf{e}} dV \right] \right]}_{\mathcal{D}} \Bigg]_0, \quad (4.2)
 \end{aligned}$$

where

$$\dot{\mathbf{e}} = \frac{1}{2\ell} \left[ \nabla \dot{\mathbf{u}} + (\nabla \dot{\mathbf{u}})^\top \right] \quad (4.3)$$

is the non-dimensional rate-of-strain tensor, and the notation  $[\![ \cdot ]\!]_0$  represents an average over a period of the oscillations with the slow time  $\tilde{\tau}$  held constant.

The terms on the right-hand side of equation (4.2) are precisely the energy budget obtained by Whittaker *et al.* [13] for constant-amplitude oscillations. The new term on the left-hand side corresponds to the rate of change of kinetic energy over the slow time scale.

We define the kinetic energy  $\mathbb{E}$ , and the fluxes  $\mathcal{K}$ ,  $\mathcal{J}$ ,  $\mathcal{E}$  and  $\mathcal{D}$  as indicated in equation (4.2).  $\mathcal{K}$  is the net flux of kinetic energy through the tube ends,  $\mathcal{J}$  is the rate of working by pressure forces at the tube ends,  $\mathcal{E}$  is the rate of working by the fluid on the tube wall and  $\mathcal{D}$  is the rate of energy loss through viscous dissipation. The energy budget can therefore be written as

$$\frac{d\mathbb{E}}{d\tilde{\tau}} = \mathcal{K} - \mathcal{J} - \mathcal{E} - \mathcal{D}. \quad (4.4)$$

As in Whittaker *et al.* [13] we also consider the modified non-oscillatory flow  $(\bar{\mathbf{u}}', \bar{p}')$ , as defined in §3*d*, and deduce that it satisfies the approximate energy budget

$$\frac{d\mathbb{E}'}{d\tilde{\tau}} = \mathcal{K}' - \mathcal{J}' - \mathcal{D}' + O(\epsilon^2, \epsilon\delta^2, \delta^4). \quad (4.5)$$

Each of the primed symbols above uses the corresponding definition of the unprimed version, but with  $(\tilde{\mathbf{u}}, \tilde{p})$  replaced by  $(\tilde{\mathbf{u}}', \epsilon \tilde{p}'/\lambda)$ .<sup>5</sup> (There is no  $\mathcal{E}'$  term in equation (4.5) since the boundary condition (3.13*d*) means that the fluid velocity is almost zero on the wall and, as a result, the work done by the fluid there is negligible.)

The fluxes  $\mathcal{K}$ ,  $\mathcal{K}'$ ,  $\mathcal{S}$ ,  $\mathcal{S}'$ ,  $\mathcal{D}$  and  $\mathcal{D}'$  are all unchanged from Whittaker *et al.* [13] at leading order, since they just involve quantities from the leading-order solution for the fluid flow. We can therefore use the results that

$$\mathcal{K} = \mathcal{K}' + \frac{3\lambda^2\delta^2\mathbb{Q}_0}{4} [|\tilde{w}_{00}|^2]_{\tilde{z}=1}^{\tilde{z}=0}, \quad (4.6a)$$

$$\mathcal{S} = \mathcal{S}' - \frac{\lambda^2\delta^2\mathbb{Q}_0}{4} [|\tilde{w}_{00}|^2]_{\tilde{z}=1}^{\tilde{z}=0} \quad (4.6b)$$

and

$$\mathcal{D} = \mathcal{D}' + \frac{\tilde{\omega}^{1/2}\lambda^3\delta^2}{2\sqrt{2}R} \int_0^1 C_0(\tilde{z})|\tilde{w}_{00}|^2 d\tilde{z}, \quad (4.6c)$$

to leading order, where  $\mathbb{Q}_0$  is the non-oscillatory component of the volume flux, and  $C_0(\tilde{z})$  is the circumference of the tube wall in the undeformed configuration.

The rate of working  $\mathcal{E}$  involves higher order terms in  $\tilde{\mathbf{u}}$  and  $\tilde{p}$ , and so is affected by the amplitude variation. (The leading-order fields for  $\tilde{\mathbf{u}}$  and  $\tilde{p}$  are 1/4-period out of phase and so average to zero, while the factor of  $1/\epsilon$  promotes the contributions from the next order down.) However, rather than evaluating  $\mathcal{E}$  directly, it is more convenient to compute  $d\mathbb{E}/d\tilde{\tau}$  and then infer the value of  $\mathcal{E}$  from the overall energy budget (4.4).

By substituting the expansions (2.16*a-c*) into the integral for  $\mathbb{E}$  in equation (4.2), we find that

$$\mathbb{E} = \frac{1}{2} \iiint_{V_0} \bar{w}^2 + \epsilon\beta|\tilde{\mathbf{u}}^\perp|^2 + \frac{1}{2}\lambda^2\delta^2|\tilde{w}_{00}|^2 dV + O(\epsilon\delta^2, \delta^4) \quad (4.7a)$$

and

$$\mathbb{E} = \mathbb{E}' + \frac{1}{2}\lambda^2\delta^2 \int_0^1 A_0(\tilde{z})|\tilde{w}_{00}|^2 d\tilde{z} + O(\epsilon\delta^2, \delta^4). \quad (4.7b)$$

Finally, substituting the expressions for the fluxes (4.6*a-c*) and energy (4.7) into the energy budget (4.4) and making use of the modified budget (4.5), we find that

$$\mathcal{E} = \lambda^2\delta^2 \left( \frac{\mathbb{Q}_0}{2} [|\tilde{w}_{00}|^2]_{\tilde{z}=1}^{\tilde{z}=0} - \frac{\tilde{\omega}^{1/2}\lambda}{2\sqrt{2}R} \int_0^1 C_0(\tilde{z})|\tilde{w}_{00}|^2 d\tilde{z} - A \int_0^1 A_0(\tilde{z})|\tilde{w}_{00}|^2 d\tilde{z} \right). \quad (4.8)$$

The energy flux to the wall is therefore given by the net flux of oscillatory kinetic energy through the tube ends, minus work done at the tube ends, minus the energy lost through dissipation, minus the kinetic energy taken up by the growth of the oscillations. Energy changes required by the temporal variation in the non-oscillatory axial flow decouple by virtue of the modified balance (4.5).

<sup>5</sup>The scale factor in the pressure here is a result of the scalings used in equation (2.16*c*).

A Reynolds number for the non-oscillatory flow can be defined in the usual way by

$$Re = \frac{\rho \mathcal{U} a}{\mu} = \frac{\alpha^2}{St}. \quad (4.9)$$

Then, using equations (4.8) and (2.9*a–d*), the condition under which energetically neutral ( $\mathcal{E} = 0$ ) constant-amplitude ( $\mathcal{A} = 0$ ) oscillations are possible can be expressed as a critical Reynolds number

$$Re_c = \frac{\alpha \ell \tilde{\omega}^{1/2}}{\sqrt{2\pi}(|\tilde{w}_{00}(0)|^2 - |\tilde{w}_{00}(1)|^2)} \int_0^1 C_0(\tilde{z}) |\tilde{w}_{00}(\tilde{z})|^2 d\tilde{z}. \quad (4.10)$$

This is the same result as derived by Whittaker *et al.* [13]. However, the full expression (4.8) allows us to quantitatively link the growth rate  $\mathcal{A}$  and the net energy transfer to the wall  $\mathcal{E}$ , for a given mode shape and mean-flow Reynolds number. For  $Re < Re_c$ , we must have a decaying amplitude and/or a net transfer of energy to the fluid from the wall, while, for  $Re > Re_c$ , we must have a growing amplitude and/or a net transfer of energy from the fluid to the wall.

## 5. Discussion

In this paper, we have studied the effect of prescribed wall oscillations with a slowly varying amplitude on flow through an elastic-walled tube. The typical growth or decay rates were chosen to be comparable with the variation that would occur in the case of self-excited oscillations. As in Whittaker *et al.* [13], in which the fixed oscillation amplitude case was studied, we worked in the small-amplitude, high-frequency, long-wavelength regime specified by equations (2.9*a–d*). The key results of this paper are the calculation of the leading- and first-order effects on the fluid flow caused by the amplitude variation, and the evaluation of the revised energy budget at leading order.

This current paper provides key results that are needed to build a model of self-excited oscillations in a collapsible-tube system. The details and solution of this model are reported elsewhere [14], but, briefly, we combine the fluid mechanics solutions from the current paper with a description of the tube-wall mechanics from Whittaker *et al.* [16]. The two are coupled via the kinematic and dynamic boundary conditions on the tube wall, and we solve the combined system to find the normal modes of oscillation. The energy calculation in the current paper is extended to include the elastic energy in the tube wall, and the overall energy budget can be used to determine growth and decay rates. It is then possible to infer the stability of the system to the various normal modes, and their associated growth rates.

In §2*a*, we estimated scales for the growth or decay rate for free oscillations in the regime (2.9*a–d*). The time scale was found to be much longer than the oscillation period, allowing a multiple-scale analysis to be used. We found in §3*b* that the growth or decay would be sufficiently slow so as not to affect the oscillatory flow in the fluid at leading order. The corrections that enter at first order are evaluated in §3*c*.

By evaluating the energy budget, we were able to derive an expression (4.8) for the leading-order time-averaged rate of working  $\mathcal{E}$  by the fluid on the tube wall, in terms of the solution  $\tilde{u}_{00}(\tilde{z})$  for the leading-order oscillatory axial flow, the non-oscillatory flux  $\mathbb{Q}_0$ , the oscillation frequency  $\tilde{\omega}$ , the growth rate  $\Lambda$  and the various dimensionless parameters. Note that only the leading-order fluid solution is required to evaluate  $\mathcal{E}$ . The critical mean-flow Reynolds number (4.10) for energetically neutral constant-amplitude oscillations matches the result derived by Whittaker *et al.* [13]. From this work, we also gain the quantitative link between the growth rate and energy transfer to the wall for a given mode shape and mean-flow Reynolds number, which was not available previously.

There is a subtlety that arises in the case where the mean flow is driven by a steady pressure difference (as opposed to an imposed volume flux at one end). The interaction of the oscillatory flow with itself in the nonlinear inertia term produces a non-oscillatory ‘Reynolds stress’ contribution. In the pressure-driven case, these Reynolds stresses induce a non-oscillatory perturbation to the steady flow, altering the non-oscillatory flow rate. This could affect the energy budget in two ways. First, the kinetic energy of the non-oscillatory flow is altered. Second, the oscillatory kinetic energy flux at the tube ends is proportional to the non-oscillatory flow rate, which has changed. Simple scaling arguments show that both of these energy changes could enter at leading order.

However, we have shown here, in §4, that the energy associated with the change in the kinetic energy of the non-oscillatory flow decouples from the main energy budget for the oscillatory flow. This effect therefore makes no contribution at all to equation (4.8).

Second, as shown in §3*d*, significant changes in the non-oscillatory flow occur only over a time scale that is much longer than both the  $O(T)$  oscillation period and the  $O(T/\epsilon)$  time scale of the amplitude growth or decay. This means that there will normally be significant growth or decay before there is a significant change to the non-oscillatory axial flow. Therefore, the change in the flux does not have a significant effect on the energy budget.

An exception to this occurs if we are very close to the neutral condition. In this case, the growth or decay time scale will be longer than its  $O(T/\epsilon)$  scale estimate. The background non-oscillatory flow may then be able to adjust significantly over the time it takes for significant growth or decay in amplitude to occur. Since the flux adjustment is always positive (i.e. an increased axial volume flux) and a certain flux needs to be exceeded for instability in the case of free oscillations, this would point towards stability being lost in a sub-critical bifurcation: the oscillation-induced increase in the steady flux could take the mean-flow rate from below the criticality threshold  $Re = Re_c$  to above it.<sup>6</sup> However, this sub critical nature is not certain as the full set of nonlinear effects has yet to be analysed. What we can say though is that there exist small-amplitude energetically neutral oscillatory solutions of fixed non-zero amplitude with  $Re \equiv \rho \mathcal{U} a / \mu < Re_c$ , where  $Re$  is the Reynolds number based on the steady flow that would result from the applied pressure difference in the absence of oscillations.

<sup>6</sup>Such sub critical behaviour was observed by Stewart *et al.* [11] for a one-dimensional model of channel flow in a slightly different regime, and also accounted for by the effect of mean-flow adjustment.

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