6 The principle of virtual displacements (PVD)

- The principle of virtual displacements is one of the most powerful theorems of continuum mechanics and is extremely useful for the efficient computational solution of large displacement elasticity problems.

- The PVD can either be 'derived' from the equilibrium equations or it can be stated axiomatically. In the latter case, the equilibrium equations follow from it and represent the Euler-Lagrange equations associated with the variational statement.

- Physically, the PVD is related to energy/work based approaches but it is not restricted to conservative loads.

- The PVD is also closely related to the Method of Weighted Residuals used in the numerical solution of PDEs.

6.1 The virtual displacement field

- Consider an elastic body whose deformed volume V occupies the region B. The interior of the body is subject to a body force $\rho\mathbf{B}$ and the part ∂B_t of its boundary is subject to an external traction ${}_o\boldsymbol{\tau}$. Displacement boundary conditions are applied along the remaining part of the boundary ∂B_u (such that $\partial B_t + \partial B_u = \partial B = A$). We assume that the body is in equilibrium (see Fig. 8).

- We will now subject this body to a small (virtual) displacement $\delta\mathbf{u}$ which is consistent with the displacement boundary conditions. The body force and the traction are held at constant values during this virtual displacement.

- Mathematically, this procedure corresponds to a variation of the displacement field and the usual rules of the calculus of

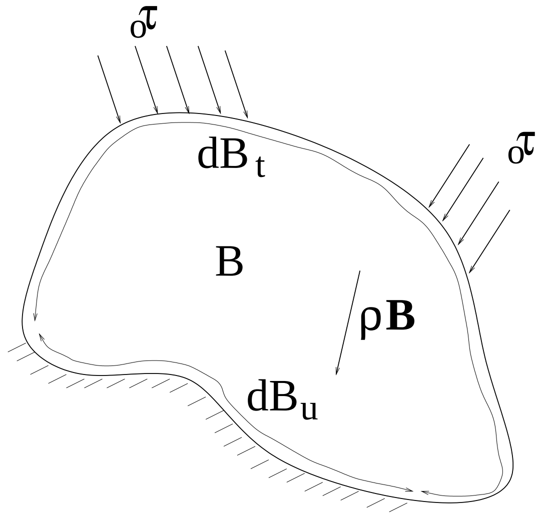

Figure 8: Sketch of a deformed body occupying the region B, subject to a body force $\rho\mathbf{B}$ and to prescribed tractions ${}_o\boldsymbol{\tau}$ along ∂B_t. Displacement boundary conditions are applied along ∂B_u.

variations apply:

- The order of variations and differentiations can be exchanged:

$$\delta\frac{\partial\mathbf{u}}{\partial x^i} = \frac{\partial\delta\mathbf{u}}{\partial x^i}.$$

- The order of variations and integrations can be exchanged:

$$\delta\int\int\int\mathbf{u}\ dB = \int\int\int\delta\mathbf{u}\ dB.$$

- Variations observe the product rule, i.e. for two functions $\mathcal{F}$ and $\mathcal{G}$ which depend on the displacement field $\mathbf{u}$ and its derivatives we have

$$\delta(\mathcal{F}\mathcal{G}) = \delta\mathcal{F}\ \mathcal{G} + \mathcal{F}\ \delta\mathcal{G}.$$

- The formal rule to obtain the variation of a function $\mathcal{F}$ which depends on the displacement field $\mathbf{u}$ and its partial derivatives, i.e.

$$\mathcal{F} = \mathcal{F}(x^i, \mathbf{u}, \mathbf{u}_{,i}, \mathbf{u}_{,ij}, ...)$$

is

$$\delta\mathcal{F} = \frac{\partial\mathcal{F}}{\partial\mathbf{u}}\delta\mathbf{u} + \frac{\partial\mathcal{F}}{\partial\mathbf{u}_{,i}}\delta\mathbf{u}_{,i} + \frac{\partial\mathcal{F}}{\partial\mathbf{u}_{,ij}}\delta\mathbf{u}_{,ij} + ...$$

- Let us now consider the following expression

$$\delta\Pi = \underbrace{\int\int\int_B \rho_{def}\mathbf{B}\cdot\delta\mathbf{u}\,dV}_{\int\int\int_B \mathbf{K}\cdot\delta\mathbf{u}\,dx^1dx^2dx^3} + \int\int_{B_t} {}_o\boldsymbol{\tau}\cdot\delta\mathbf{u}\,dA$$

which represents the work done by the body force $\rho\mathbf{B}$ and the applied traction ${}_o\boldsymbol{\tau}$ when the body is subjected to the virtual displacement $\delta\mathbf{u}$.

- We use the compact form (16) of the equilibrium equations to replace the body force $\mathbf{K}$ term by $-\mathbf{T}^i_{,i}$ and prepare for an integration by parts by writing

$$\mathbf{T}^i_{,i}\cdot\delta\mathbf{u} = \left(\mathbf{T}^i\cdot\delta\mathbf{u}\right)_{,i} - \left(\mathbf{T}^i\cdot\delta\mathbf{u}_{,i}\right).$$

- This yields

$$\begin{aligned}\delta\Pi = & - \int\int\int\left(\mathbf{T}^i\cdot\delta\mathbf{u}\right)_{,i}\,dx^1dx^2dx^3 \\ & + \int\int\int\left(\mathbf{T}^i\cdot\delta\mathbf{u}_{,i}\right)\,dx^1dx^2dx^3 + \\ & + \int\int_{B_t} {}_o\boldsymbol{\tau}\cdot\delta\mathbf{u}\,dA\end{aligned}$$

- Now we use Gauss' theorem (8) and the definition of $\mathbf{T}^i$ in terms of the second Piola Kirchhoff stress tensor (see (17)) to transform the first integral into

$$-\int\int\int\left(\mathbf{T}^i\cdot\delta\mathbf{u}\right)_{,i}\,dx^1dx^2dx^3 = -\int\int\underbrace{\sigma^{ij}\mathbf{G}_j n_i}_{\boldsymbol{\sigma}}\cdot\delta\mathbf{u}\,da,$$

where we recognise the Piola Kirchhoff stress vector $\boldsymbol{\sigma}$ (the force referred to the undeformed area element). Note that the integration has to be carried out over the undeformed area.

- Since ${}_o\boldsymbol{\tau}\,dA = {}_o\boldsymbol{\sigma}\,da$, we can combine both surface integrals to integrals over the undeformed surface and obtain

$$\delta\Pi = \int\int\int \mathbf{T}^i\cdot\delta\mathbf{u}_{,i}\,dx^1dx^2dx^3 - \int\int(\boldsymbol{\sigma} - {}_o\boldsymbol{\sigma})\cdot\delta\mathbf{u}\,da,$$

which shows that the surface integral vanishes because of the traction boundary condition, $\boldsymbol{\sigma} - {}_o\boldsymbol{\sigma} = \mathbf{0}$.

- Now we express $\mathbf{T}^i$ in terms of the second Piola Kirchhoff stress tensor (see (17)) and obtain

$$\delta\Pi = \int\int\int \underbrace{\sigma^{ij}\mathbf{G}_j \cdot \delta\mathbf{u}_{,i}}_{\sigma^{ij}\ \delta\epsilon_{ij}} \ \underbrace{\sqrt{g}\ dx^1 dx^2 dx^3}_{dv},$$

where we can identify the undeformed volume element dv. It is left as an exercise to show that $\sigma^{ij}\mathbf{G}_j \cdot \delta\mathbf{u}_{,i} = \sigma^{ij}\delta\epsilon_{ij}$ (use the definition of the deformed basis vector (see (9)) and the symmetry of σ^{ij}).

- Finally, equating the two representations of $\delta\Pi$, we obtain the PVD

$$\int\int\int_B \sigma^{ij}\delta\epsilon_{ij} dv = \int\int\int_B \rho_{def}\mathbf{B} \cdot \delta\mathbf{u}\ dV + \int\int_{\partial B_t} {}_o\boldsymbol{\tau} \cdot \delta\mathbf{u}\ dA$$

- Formally, the PVD in the large displacement regime has exactly the same form as its counterpart in linear elasticity. However, only the above derivation shows which stress tensor is the appropriate (work conjugate) choice for the strain description based on the non-linear strain tensor ϵ_{ij}.

- Note that we can rewrite the load terms on the right hand side in the most 'convenient' basis (undeformed or deformed), depending on the character of the loads (spatially fixed or follower loads). Hence an alternative form of the PVD is

$$\int\int\int_B \sigma^{ij}\delta\epsilon_{ij} dv = \int\int\int_B \rho\mathbf{b} \cdot \delta\mathbf{u}\ dv + \int\int_{\partial B_t} {}_o\boldsymbol{\sigma} \cdot \delta\mathbf{u}\ da.$$

In this form all quantities are referred to the undeformed configuration.

- The physical interpretation of the PVD is as follows: During a virtual displacement about an equilibrium position, the (virtual) work done by the external loads (body force and traction) is equal to the work done by the internal stresses. These stresses may or may not be conservative; the PVD is a general continuum mechanical statement, it is not restricted to elastic bodies!

6.2 The relation of the PVD to other formulations

6.2.1 Energy based approach for elastic bodies

- We know from the physical interpretation of the PVD that $\int\int\int_B \sigma^{ij}\delta\epsilon_{ij}dv$ represents the work done by the internal stresses during the virtual displacement. Provided the body is elastic, this must be equal to the variation in the body's strain energy, i.e.

$$\int\int\int_B \sigma^{ij}\ \delta\epsilon_{ij}\ dv = \delta\int\int\int w\ dv \tag{23}$$

where $w(\epsilon_{ij})$ is the strain energy per unit undeformed volume of the body, as defined in (22).

- Now we use the rules for the calculus of variations and carry out the variation on the right hand side of (23) with respect to the strain tensor. Thus we obtain

$$\delta\int\int\int w\ dv = \int\int\int \delta w\ dv = \int\int\int \frac{\partial w}{\partial \epsilon_{ij}}\ \delta\epsilon_{ij}\ dv. \tag{24}$$

Comparing (23) and (24), we see that

$$\sigma^{ij} = \frac{\partial w}{\partial \epsilon_{ij}},$$

which shows that the second Piola Kirchhoff stress σ^{ij} is indeed the 'work conjugate' stress measure for the strain tensor ϵ_{ij}, as claimed previously.

6.2.2 Minimal potential energy

- Let us now consider the case of an elastic body which is subject to forces (traction and body force) which are conservative and can therefore be derived from displacement potentials φ_b and φ_τ, respectively, i.e.

$$_o\boldsymbol{\tau} = -\frac{\partial \varphi_\tau}{\partial \mathbf{u}} \quad \text{and} \quad \rho \mathbf{b} = -\frac{\partial \varphi_b}{\partial \mathbf{u}}$$

- In this case, we can re-write the PVD as

$$\delta \int\int\int (w + \varphi_b + \varphi_\tau) dv = 0.$$

This is the theorem of minimum potential energy: The equilibrium states of an elastic body subject to conservative loads are characterised by an extremum of the total potential energy $\int\int\int (w + \varphi_b + \varphi_\tau) dv$. Equilibrium states for which the total potential energy is a minimum (maximum) can be shown to be stable (unstable).

6.2.3 The equilibrium equations

- The equilibrium equations are the Euler Lagrange equations associated with the variational principle. To derive them from the PVD, we first decompose the displacements and the applied forces and tractions into the undeformed basis (say), i.e.

$$\mathbf{b} = b^i \mathbf{g}_i = b_i \mathbf{g}^i,$$

$$_o\boldsymbol{\sigma} = {}_o\sigma^i \, \mathbf{g}_i = {}_o\sigma_i \, \mathbf{g}^i,$$

and

$$\mathbf{u} = u^i \mathbf{g}_i = u_i \mathbf{g}^i.$$

- In order to derive the equilibrium equations in terms of the displacement field, we need to carry out the variations of all quantities with respect to displacement components and their partial derivatives with respect to the Lagrangian coordinates, i.e.

$$\delta \mathbf{u} = \delta u^i \, \mathbf{g}_i = \delta u_i \, \mathbf{g}^i$$

and

$$\delta \epsilon_{ij} = \frac{\partial \epsilon_{ij}}{\partial u^k} \, \delta u^k + \frac{\partial \epsilon_{ij}}{\partial u^k_{,l}} \, \delta u^k_{,l}$$

- Inserting this into the PVD, we obtain

$$\int\!\!\int\!\!\int \left\{ \sigma^{ij} \left(\frac{\partial \epsilon_{ij}}{\partial u^k} \, \delta u^k + \underline{\frac{\partial \epsilon_{ij}}{\partial u^k_{,l}} \, \delta u^k_{,l}} \right) - \rho b_i \, \delta u^i \right\} dv - \int\!\!\int {}_o\sigma_i \, \delta u^i \, da = 0.$$

- Now we integrate the underlined term in the volume integral by parts and apply the divergence theorem (8) to obtain

$$\int\!\!\int\!\!\int \left\{ \sigma^{ij} \frac{\partial \epsilon_{ij}}{\partial u^k} - \left(\sigma^{ij} \frac{\partial \epsilon_{ij}}{\partial u^k_{,l}} \right)_{,l} - \rho b_k \right\} \delta u^k \, dv + \int\!\!\int (...) \delta u^k \, da = 0, \tag{25}$$

where the surface integral contains contributions from the applied traction ${}_o\sigma_i$ and from the surface integral generated by the application of the divergence theorem.

- Note that we have not placed any restrictions on the virtual displacement field δu^k in the interior of the body. Hence, the volume integral can only be zero (for all choices of δu^k) if the terms in the curly brackets vanish. This yields the three partial differential equations

$$\mathcal{L}(\mathbf{u}) = \sigma^{ij} \frac{\partial \epsilon_{ij}}{\partial u^k} - \left(\sigma^{ij} \frac{\partial \epsilon_{ij}}{\partial u^k_{,l}} \right)_{,l} - \rho b_k = 0, \tag{26}$$

which represent the equilibrium equations.

- Similarly, the terms in the round brackets in the surface integral have to vanish along the parts of the boundary ∂B_t

on which no displacement boundary conditions are applied. They represent the stress boundary conditions and appear as the natural boundary condition from the variational principle.

- Finally, we had assumed that the displacement field fulfills the essential (displacement) boundary conditions on the remaining parts of the boundary, ∂B_u, and that these boundary conditions are not affected by the application of the virtual displacement. Hence, we have $\delta u^k = 0$ on ∂B_u which implies that the surface integral over ∂B_u vanishes automatically.

6.2.4 Weak solutions and the method of weighted residuals

- The derivation of the equilibrium equations from the variational statement shows the close connection between the variational statement and the concept of a weak solution which is exploited in many numerical solution techniques – usually summarised under the name 'method of weighted residuals (MWR)'.
- The idea behind a weak solution of a differential equation $\mathcal{L}(\mathbf{u}) = 0$ is as follows. Rather than fulfilling the equation pointwise, we require it to be fulfilled in a weighted average sense

$$\int\int\int \mathcal{L}(\mathbf{u})\psi dv = 0. \tag{27}$$

- If we can show that for a certain function $\mathbf{u} = \hat{\mathbf{u}}$, this equation is fulfilled for *any* choice of the weighting function ψ, then we call $\hat{\mathbf{u}}$ a weak solution.
- The numerical implementation of this approach leads to the Galerkin method.
- Comparing (27) to (25) and (26) shows that the virtual displacement $\delta\mathbf{u}$ in the PVD plays the role of weighting function

ψ in the MWR.

- Interestingly, in most applications of Galerkin's method, one tries to reduce the differentiability requirements to be placed on the displacements and the weighting function, by integrating (27) by parts – thus essentially reversing the derivation of the equilibrium equations from the variational statement.

- The PVD provides a formulation in which the minimal differentiability requirements are already achieved: Only the first derivatives of the displacement and the weighting function are required (as opposed to second derivatives of the displacements in the equilibrium equations).