5 The Constitutive Equations

- So far, we have described the kinematics of the deformation by expressing the strain tensor as a function of the displacement field, i.e.
$$\epsilon_{ij} = \epsilon_{ij}(u^k)$$
and we have formulated the equilibrium equations in terms of the stress tensor, i.e.
$$\tau^{ij}||_j + \rho B^i = 0.$$

- These equations are incomplete: we still have to specify the relation between the stress and strain tensors, i.e. the *constitutive equations* in the form
$$\tau^{ij} = \tau^{ij}(\epsilon_{kl}) \tag{20}$$
or
$$\sigma^{ij} = \sigma^{ij}(\epsilon_{kl}). \tag{21}$$

- Note that the form of the constitutive equations postulated in (20) and (21) is fairly restrictive as it excludes viscous or viscoelastic behaviour which is characterised by constitutive equations of the form $\tau^{ij} = \tau^{ij}(\dot{\epsilon_{kl}})$ and $\tau^{ij} = \tau^{ij}(\epsilon_{kl}, \dot{\epsilon_{kl}})$, respectively.

- While all derivations in the previous sections were valid for arbitrary material behaviour, we will now restrict ourselves to *elastic materials.* Thus, we assume that
$$\tau^{ij}(x^k, t, \epsilon_{lm}) = \tau^{ij}(\epsilon_{lm}(x^k, t)),$$
i.e. we assume that the local stress depends *only* on the *local, instantaneous* value of the strain.

- In practice, the determination of constitutive equations requires laboratory experiments and a certain amount of curve fitting. However, in many cases we can at least determine the appropriate mathematical form of the constitutive equations.

5.1 Small strain

- For bodies subject to small strain, we choose the reference configuration as the undeformed, stress-free state such that

$$\tau^{ij}(\epsilon_{kl} = 0) = 0.$$

- If we subject the body to a deformation which induces small strains, i.e. $\epsilon_{kl} = \mathcal{O}(\epsilon) \ll 1$ we can (in principle) Taylor expand the (unknown!) constitutive equation about the stress-free reference state and obtain

$$\tau^{ij}(\epsilon_{kl}) = \underbrace{\left.\frac{\partial \tau^{ij}}{\partial \epsilon_{kl}}\right|_{\epsilon_{kl}=0}}_{E^{ijkl}} \epsilon_{kl} + \mathcal{O}(\epsilon^2).$$

- E^{ijkl} is the fourth order tensor of elastic constants which has $3^4 = 81$ coefficients. Symmetry considerations show that only 21 of these coefficients are independent and have to be determined experimentally.

- For an isotropic body there are only two free constants which are known as the Lamé coefficients, λ and μ, and the elasticity tensor is given by the isotropic fourth order tensor in the deformed configuration

$$E^{ijkl} = \lambda G^{ij} G^{kl} + \mu (G^{ik} G^{jl} + G^{il} G^{jk}).$$

Since we are dealing with small strains, we can replace the deformed metric tensor G^{ij} by the undeformed metric tensor

g^{ij} without introducing any additional errors, i.e.

$$E^{ijkl} = \lambda g^{ij} g^{kl} + \mu (g^{ik} g^{jl} + g^{il} g^{jk}).$$

5.2 Small strains superimposed on finite deformations

- Assume that a body has been subjected to a (known) finite deformation which induces stresses τ_0^{ij}.
- We choose this configuration (in which the body is subject to initial stresses τ_0^{ij}) as the reference configuration and Taylor expand the stresses with respect to the small strains $\epsilon_{kl} = \mathcal{O}(\epsilon) \ll 1$ which are induced by the superimposed deformation, i.e.

$$\tau^{ij}(\epsilon_{kl}) = \tau_0^{ij} + \underbrace{\left.\frac{\partial \tau^{ij}}{\partial \epsilon_{kl}}\right|_{\epsilon_{kl}=0}}_{\hat{E}^{ijkl}} \epsilon_{kl} + \mathcal{O}(\epsilon^2).$$

- The fourth order tensor $\hat{E}^{ijkl}$ represents the 'incremental stiffness' of the body about the initial (pre-deformed or pre-stressed state).
- Note that this formulation also allows the description of a body which is subject to initial stresses which were generated by a different mechanism (e.g. thermal stresses).

> Note that small strain does not necessarily imply small deformations! For instance, thin walled shell structures can undergo very large deformations without inducing large strains.

5.3 Large strain

- Many materials of practical importance (e.g. rubber) can undergo large strains while still behaving elastically. Most of these materials are incompressible.
- Large-strain constitutive equations tend to be highly non-linear and few general statements about the 'appropriate form' of the constitutive equations can be made.
- One of the most popular approaches is to postulate equations for the strain energy W (i.e. the energy per undeformed volume, stored in the elastic body during the deformation), i.e.

$$w = w(\epsilon_{ij}). \tag{22}$$

From this expression, the appropriate ('work conjugate') stresses can be derived from

$$\sigma^{ij} = \frac{\partial w}{\partial \epsilon_{ij}}.$$

We will return to this relation in the next section.

- The strain energy is often expressed in terms of the invariants I_1, I_2 and I_3 of the strain tensor, i.e.

$$w = w(I_1, I_2, I_3)$$

where

$$I_1 = g^{ij} G_{ij}, \quad I_2 = g_{ij} G^{ij} I_3 \quad \text{and} \quad I_3 = \sqrt{G/g}.$$

- 'Popular' examples for such strain energy functions are

Mooney-Rivlin materials

$$w = C_1(I_1 - 3) + C_2(I_2 - 3).$$

Neo-Hookean materials

$$w = C_1(I_1 - 3).$$

C_1 and C_2 are constants to be determined experimentally and incompressibility $(I_3 = 1)$ is assumed in both cases.