4 Large Displacement Stress Tensors and the Equilibrium Equations

4.1 The Cauchy stress tensor

- Main difference to linear (small displacement) elasticity: We need to consider equilibrium in the deformed configuration.
- Consider an infinitesimal material tetrahedron in the deformed configuration:

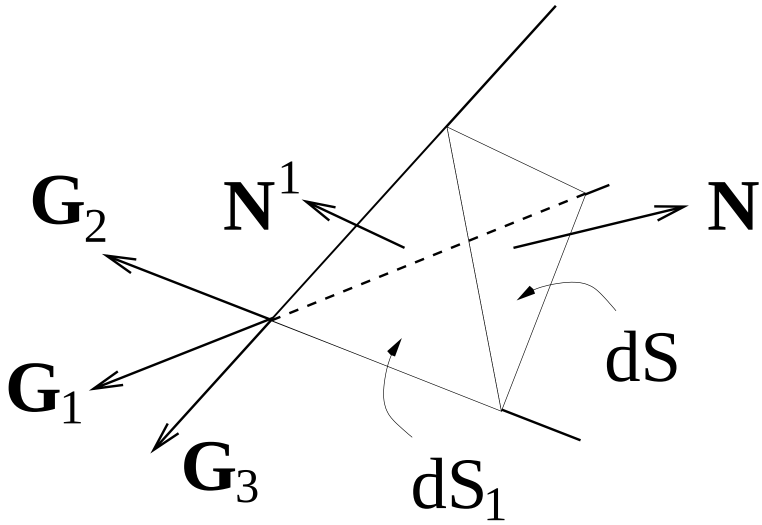

Figure 4: Sketch of an infinitesimal material tetrahedron in the deformed body. The unit normals on faces 2 and 3 are not shown.

- Elementary vector algebra for a material tetrahedron in the deformed configuration (Fig. 4):

$$\mathbf{N} dS + \mathbf{N}^i dS_i = \mathbf{0},$$

where $\mathbf{N}^i$ is unit normal on the face of the tetrahedron on which $x^i = const.$ dS_i is the scalar area of this face. $\mathbf{N}$ and dS are the corresponding quantities on the general fourth face of the tetrahedron.

- Express the face normals in terms of the normalised deformed contravariant base vectors:

$$\mathbf{N}^i = \frac{\mathbf{G}^i}{\sqrt{G^{(ii)}}}$$

- Force equilibrium on the tetrahedron:

$$\boldsymbol{\tau} dS + \boldsymbol{\tau}^i dS_i = \mathbf{0},$$

where $\boldsymbol{\tau}^i$ is the stress vector (force per unit deformed area, acting in the deformed configuration) on the face on which $x^i = const.$ $\boldsymbol{\tau}$ is the stress vector on the general fourth face of the tetrahedron.

- It follows that

$$\boldsymbol{\tau} = \boldsymbol{\tau}^i \sqrt{G^{(ii)}} N_i$$

which shows that $\boldsymbol{\tau}^i \sqrt{G^{(ii)}}$ is a contravariant quantity (this is why we chose a raised index in $\boldsymbol{\tau}^i$; note that $\boldsymbol{\tau}^i$ itself is *not* a contravariant quantity!).

- Hence, there exists a tensor τ^{ij} which decomposes $\boldsymbol{\tau}^i \sqrt{G^{(ii)}}$ into the deformed basis via

$$\boldsymbol{\tau}^i \sqrt{G^{(ii)}} =: \tau^{ij} \mathbf{G}_j.$$

- This shows that

$$\boldsymbol{\tau} = \tau^{ij} \mathbf{G}_j N_i$$

where τ^{ij} is the Cauchy stress tensor. It decomposes the stress vector $\boldsymbol{\tau}$, acting in the deformed body, in a plane with the deformed unit normal $\mathbf{N} = N_i \mathbf{G}^i$, into the deformed basis.

- If we decompose the stress $\boldsymbol{\tau}$ itself into the deformed basis, $\boldsymbol{\tau} = \tau^i \mathbf{G}_i$, we obtain the familiar looking formula

$$\tau^i = \tau^{ji} N_j.$$

4.2 The equilibrium equations

- Now consider the equilibrium of forces on an infinitesimal material block in the deformed configuration (Fig. 5)

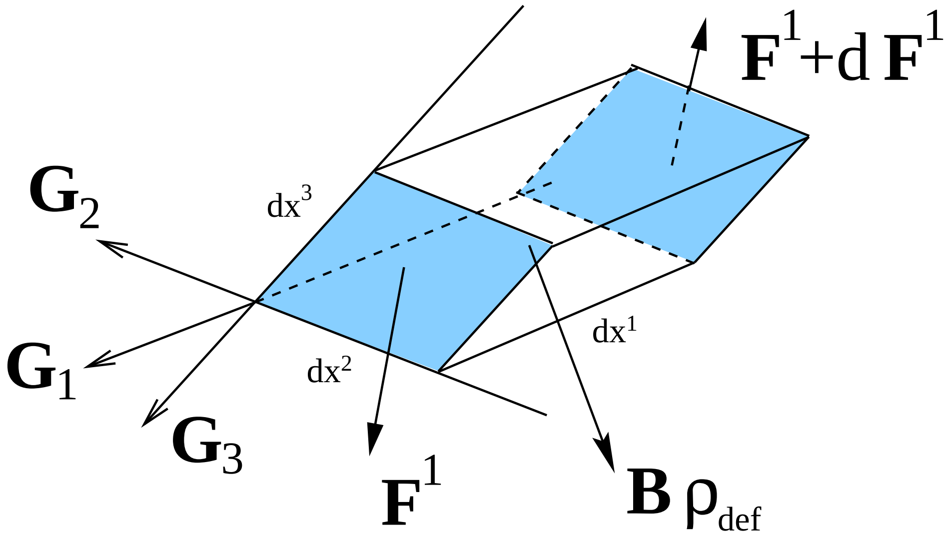

Figure 5: Sketch of an infinitesimal material block in the deformed configuration. The force transmitted through the (shaded) face on which $x^1 = const.$ is denoted by F^1 (the forces acting on the faces 2 and 3 are not shown). The body force $\mathbf{B}\rho_{def}$ acts at the centre of mass of the material block.

- The force transmitted through the face on which $x^1 = const.$ is given by $\mathbf{F}^1 = \boldsymbol{\tau}^1 dS_1$, where

$$\mathbf{F}^1 = \boldsymbol{\tau}^1 \sqrt{G^{11}G}\ dx^2 dx^3 := \mathbf{T}^1 dx^2 dx^3.$$

Equivalent equations hold for $i = 2$ and 3.

- $\mathbf{T}^i$ is the 'force per unit coordinate area' transmitted through the face on which $x^i = const.$ and is given by

$$\mathbf{T}^i = \boldsymbol{\tau}^i \sqrt{G^{(ii)}G} = \sqrt{G}\ \tau^{ij}\mathbf{G}_j. \tag{14}$$

- Equilibrium of forces implies

$$\mathbf{F}^i_{,i}\ dx^i + \mathbf{B}\ \rho_{def}\ dV = \ddot{\mathbf{u}}\ \rho_{def}\ dV \tag{15}$$

(sum over the triple index in the first term). $\mathbf{B}$ is the body force (per unit mass) and $\ddot{\mathbf{u}}$ is the acceleration of the material block. Note that there are no convective inertia terms on the right hand side of this equation since we are using Lagrangian coordinates and therefore always follow the same material particles.

- Note that conservation of mass implies

$$dm = \rho \; dv = \rho_{def} \; dV = dM$$

where $dv = \sqrt{g} \; dx^1 dx^2 dx^3$ and $dV = \sqrt{G} \; dx^1 dx^2 dx^3$ are the undeformed and deformed volume elements, respectively. Therefore, there is no difference between the body forces (per unit mass) $\mathbf{B}$ and $\mathbf{b}$ in the deformed and undeformed configuration,

$$\mathbf{B} = \mathbf{b}.$$

- Furthermore, we can replace $\rho_{def} \; dV$ by $\rho \; dv$ whenever this is more convenient (ρ and ρ_{def} are the densities in the undeformed and deformed configuration, respectively).

- D'Alembert's principle is based on the observation that in equation (15), the body force $\mathbf{B}$ and the acceleration $\ddot{\mathbf{u}}$ appear in the same form. This allows us to set the inertia terms to zero and reintroduce them (when dynamic effects are present) by replacing $\mathbf{B}$ by $\mathbf{B} - \ddot{\mathbf{u}}$.

- Similar to the definition of $\mathbf{T}^i$ we now define a 'body force per coordinate volume'

$$\mathbf{K} := \mathbf{B} \; \rho_{def} \; \sqrt{G} = \mathbf{b} \; \rho \; \sqrt{g}.$$

- This allows us to write the equilibrium equations in compact form as

$$\mathbf{T}^i_{,i} + \mathbf{K} = \mathbf{0}.$$

- Expressing $\mathbf{T}^i$ in terms of the stress tensor yields

$$\left(\sqrt{G}\ \tau^{ij}\mathbf{G}_j\right)_{,i} + \mathbf{K} = \mathbf{0}.$$

Carrying out a bit of tensor analysis (see e.g. Klingbeil 1989, p.136) transforms this equation into

$$\tau^{ij}||_j + \rho_{def}\ B^i = 0,$$

where we have decomposed the body force into the deformed basis, $\mathbf{B} = B^i\mathbf{G}_i$. $(.)||_j$ represents the covariant derivative in the deformed configuration:

$$\tau^{ij}||_k = \tau^{ij}_{,k} + \hat{\Gamma}^i_{km}\tau^{mj} + \hat{\Gamma}^j_{km}\tau^{im},$$

where the $\hat{\Gamma}^i_{km}$ are the deformed Christoffel symbols

$$\hat{\Gamma}^k_{ij} = \mathbf{G}_{i,j} \cdot \mathbf{G}^k.$$

- Note that we obtain the equilibrium equations of linear elasticity (i.e. the equilibrium equations in the undeformed configuration) if we replace the $\mathbf{G}_i$ by $\mathbf{g}_i$ and the deformed Christoffel symbols by the undeformed ones.

- Now consider the equilibrium of moments of the tractions acting on the (2D) material block shown in Fig. 6 and note that at leading order the tractions act at the center of the faces and the body force does not give a contribution. Hence we have

$$\begin{aligned}\mathbf{0} &= \left(\mathbf{G}_1 dx^1 + \mathbf{G}_2\frac{dx^2}{2}\right) \times \mathbf{T}^1 dx^2 - \mathbf{G}_2\frac{dx^2}{2} \times \mathbf{T}^1 dx^2 + \\ &+ \left(\mathbf{G}_1\frac{dx^1}{2} + \mathbf{G}_2 dx^2\right) \times \mathbf{T}^2 dx^1 - \mathbf{G}_1\frac{dx^2}{2} \times \mathbf{T}^2 dx^1.\end{aligned}$$

- Similarly the equilibrium of moments in 3D requires

$$\mathbf{0} = \mathbf{G}_i \times \mathbf{T}^i.$$

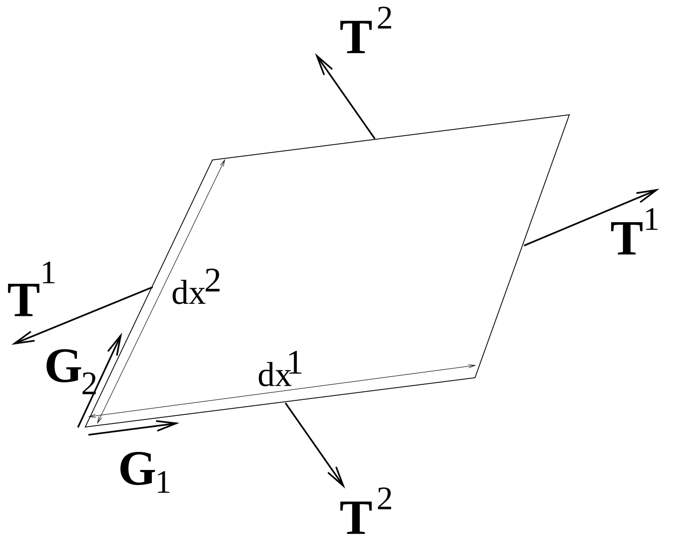

Figure 6: Equilibrium of moments on an infinitesimal block of material.

- Using $\mathbf{T}^i = \sqrt{G}\, \tau^{ij} \mathbf{G}_j$, we obtain

$$\sqrt{G}\, \tau^{ij} \left(\mathbf{G}_i \times \mathbf{G}_j \right) = \mathbf{0}.$$

- The orthogonality of the co- and contravariant base vectors implies that

$$\mathbf{G}_1 \times \mathbf{G}_2 = \sqrt{G}\, \mathbf{G}^3 \quad \text{etc.}$$

Therefore

$$\left(\tau^{12} - \tau^{21} \right) \mathbf{G}^3 + \left(\tau^{23} - \tau^{32} \right) \mathbf{G}^1 + \left(\tau^{31} - \tau^{13} \right) \mathbf{G}^2 = \mathbf{0},$$

and since the base vectors are linearly independent, this can only be fulfilled if

$$\tau^{ij} = \tau^{ji}.$$

Hence the Cauchy stress tensor is symmetric.

4.3 The stress boundary conditions

- Assume that a part ∂B_t of the surface ∂B of the deformed body is subject to an applied traction (force per unit deformed area) ${}_0\boldsymbol{\tau}$.

- Then the stress boundary condition states that ${}_o\boldsymbol{\tau}$ has to be equal to the (internal) stress at the surface of the body, i.e.

$$ {}_o\boldsymbol{\tau} = \boldsymbol{\tau}\big|_{\partial B_t} . $$

- We decompose the applied traction into the deformed basis, ${}_o\boldsymbol{\tau} = {}_o\tau^i\, \mathbf{G}_i$, and obtain the stress boundary condition

$$ {}_o\tau^i = \tau^{ji} N_j = \tau^{ij} N_j \quad \text{on } \partial B_t, $$

where the N_i are the covariant components of the unit normal $\mathbf{N}$ on the deformed surface, i.e. $\mathbf{N} = N_i \mathbf{G}^i$.

4.4 The 2nd Piola Kirchhoff stress tensor

- The Cauchy stress $\boldsymbol{\tau}$ is a stress vector defined as a force per unit area of the deformed configuration, i.e.

$$ \boldsymbol{\tau} = \lim_{\mathcal{A}\to 0} \frac{\mathbf{F}}{\mathcal{A}} $$

- The Cauchy stress tensor τ^{ij} decomposes this vector into the deformed basis.

- Other stress tensors can be defined by referring forces to the undeformed area and/or decomposing the stress vector into the undeformed rather than the deformed basis.

- We will now consider a stress definition, which refers all forces (which act on area elements in the deformed configuration!) to the corresponding areas in the *undeformed* configuration, i.e.

$$ \boldsymbol{\sigma} = \lim_{a\to 0} \frac{\mathbf{F}}{a} $$

We will then decompose this stress vector into the *deformed* basis. This approach leads to the *2nd Piola Kirchhoff stress tensor.*

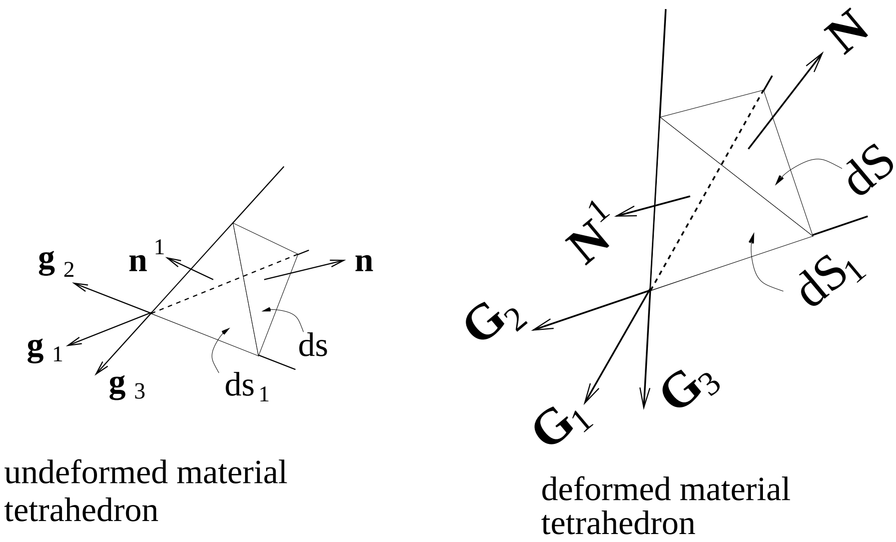

Figure 7: Sketch of an infinitesimal material tetrahedron in the undeformed and deformed configuration. (The normals on the faces 2 and 3 are not shown).

- The derivation closely follows the derivation of the Cauchy stress tensor but makes reference to the two corresponding tetrahedra in the undeformed and the deformed configuration as illustrated in Fig. 7.

- An expression for the undeformed area elements is derived by considering the undeformed tetrahedron:

$$\mathbf{n}ds + \mathbf{n}^i ds_i = \mathbf{0},$$

where $\mathbf{n}^i$ is unit normal on the face of the undeformed tetrahedron on which $x^i = const.$ ds_i is the scalar area of this face. $\mathbf{n}$ and ds are the corresponding quantities on the general fourth face of the undeformed tetrahedron.

- The equilibrium of forces implies:

$$\boldsymbol{\sigma} ds + \boldsymbol{\sigma}^i ds_i = \mathbf{0},$$

where $\boldsymbol{\sigma}^i$ is the stress vector (force per unit undeformed area, acting in the deformed configuration) on the face on which $x^i = const.$ $\boldsymbol{\sigma}$ is the stress vector on the general fourth face of the tetrahedron.

- The stress on the general face (characterised by its undeformed unit normal $\mathbf{n} = n_i \mathbf{g}^i$) is therefore given by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^i \sqrt{g^{(ii)}} n_i.$$

- Now we exploit the contravariant character of $\boldsymbol{\sigma}^i \sqrt{g^{(ii)}}$ and decompose it into the deformed basis via

$$\boldsymbol{\sigma}^i \sqrt{g^{(ii)}} := \sigma^{ij} \mathbf{G}_j$$

where σ^{ij} is the *2nd Piola Kirchhoff stress tensor.*

- This provides an equation for the stress vector in a (deformed) plane which is characterised by the unit normal $\mathbf{n} = n_i \mathbf{g}^i$ in the undeformed state:

$$\boldsymbol{\sigma} = \sigma^{ij} \mathbf{G}_j n_i.$$

- On decomposing the stress vector itself into the deformed basis, $\boldsymbol{\sigma} = \sigma^i \mathbf{G}_i$, we obtain

$$\sigma^i = \sigma^{ji} n_j.$$

- In terms of the 2nd Piola Kirchhoff stress tensor, the force transmitted over a deformed area element on which $x^1 = const.$ is given by

$$\mathbf{F}^1 = \boldsymbol{\sigma}^1 \sqrt{g^{11} g}\ dx^2 dx^3 := \mathbf{T}^1\ dx^2 dx^3$$

with similar equations for $i = 2$ and 3.

- The 'force per unit coordinate area' transmitted over an area element on which $x^i = const.$, expressed in terms of the 2nd Piola Kirchhoff stress tensor, is given by

$$\mathbf{T}^i = \boldsymbol{\sigma}^i \sqrt{g^{(ii)} g} = \sqrt{g}\ \sigma^{ij} \mathbf{G}_j. \qquad (16)$$

- Equating the two expressions for $\mathbf{T}^i$ in equations (14) and (16) shows that the two stress tensors are related by

$$\sigma^{ij} = \sqrt{\frac{G}{g}}\,\tau^{ij}$$

where $\sqrt{G/g} = dV/dv$ is the dilation of the material.

- For an incompressible material or for a body undergoing an isochoric deformation we have $G = g$ and therefore $\sigma^{ij} = \tau^{ij}$.

- Note that the symmetry of the Cauchy stress tensor implies the symmetry of the 2nd Piola Kirchhoff stress tensor, i.e.

$$\sigma^{ij} = \sigma^{ji}$$

- The equilibrium equations in terms of the 2nd Piola Kirchhoff stress tensor are given by

$$\left(\sqrt{g}\,\sigma^{ij}\mathbf{G}_j\right)_{,i} + \mathbf{K} = \mathbf{0}.$$

Since this expression involves quantities from the deformed and undeformed metric, it cannot be simplified further.

- Two further definitions for the stress tensor are conceivable:

  - force per unit deformed area, decomposed into the undeformed basis

and

  - force per unit undeformed area decomposed into the undeformed basis.

However, these definitions lead to non-symmetric stress tensors and are less useful.

4.5 The physical components of the stress tensor

- In a plane which is characterised by its deformed unit normal, $\mathbf{N}$, the Cauchy stress tensor τ^{ij} provides a linear relation between the contravariant stress components τ^i and the covariant components of the unit normal, N_j,

$$\tau^i = \tau^{ji} N_j = \tau^{ij} N_j. \tag{17}$$

- Since the deformed basis vectors are generally not unit vectors, the N_j and τ^i in this relation are not physical components.
- We will now derive the relation between the physical components N_j^* and τ^{*i} which will suggest a definition for the physical components of the stress tensor.
- We express $\mathbf{N}$ and $\boldsymbol{\tau}$ in physical components via

$$\mathbf{N} = N_j \mathbf{G}^j = N_j^* \frac{\mathbf{G}^j}{\sqrt{G^{(jj)}}} \Longrightarrow N_j = \frac{N_j^*}{\sqrt{G^{(jj)}}}$$

and

$$\boldsymbol{\tau} = \tau^i \mathbf{G}_i = \tau^{*i} \frac{\mathbf{G}_i}{\sqrt{G_{(ii)}}} \Longrightarrow \tau^i = \frac{\tau^{*i}}{\sqrt{G_{(ii)}}}.$$

- Inserting these expressions into equation (17) yields

$$\tau^{*i} = \left(\sqrt{\frac{G_{(ii)}}{G^{(jj)}}}\ \tau^{ij} \right) N_j^* =: \tau^{*ij} N_j^*$$

and we refer to

$$\tau^{*ij} = \sqrt{\frac{G_{(ii)}}{G^{(jj)}}}\ \tau^{ij}$$

as the physical components of the stress tensor. It should be noted that τ^{*ij} does not have tensor character.

- Setting $N_1^* = 1$ and $N_2^* = N_3^* = 0$ (e.g.) identifies τ^{*ij} as the component of the stress vector in the direction of the i-th covariant deformed basis vector in a plane on which $x^j = const.$

4.6 The physical components of the strain tensor

- We showed earlier that Green's strain tensor is a measure of the change of the squares of line elements during the deformation:

$$\frac{1}{2}(dS^2 - ds^2) = \epsilon_{ij}\ dx^i dx^j. \tag{18}$$

- The quadratic form on the right hand side of this equation involves differentials of the Lagrangian coordinates which are related to differential line elements via

$$d\mathbf{r} = dx^i\ \mathbf{g}_i = dx^{*i}\ \frac{\mathbf{g}_i}{\sqrt{g_{(ii)}}}$$

- We rewrite equation (18) in terms of the physical components of the Lagrangian coordinates, $dx^{*i} = dx^i\sqrt{g_{(ii)}}$, and obtain

$$\frac{1}{2}(dS^2 - ds^2) = \left(\frac{\epsilon_{ij}}{\sqrt{g_{(ii)}g_{(jj)}}}\right) dx^{*i}dx^{*j} =: \epsilon_{ij}^* dx^{*i}dx^{*j},$$

  which suggests regarding

$$\epsilon_{ij}^* = \frac{\epsilon_{ij}}{\sqrt{g_{(ii)}g_{(jj)}}}$$

  as the physical components of the strain tensor.