Non-Linear Elasticity and Computational Solid Mechanics

A Postgraduate Lecture

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1 References

Over the years I've found the following books/papers very useful

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2 Conventions/Notation

- Lower (upper) case letters for variables associated with the undeformed (deformed) body.
- Index notation.
- Summation convention but no summation over repeated indices in brackets.
- Subscript comma denotes partial differentiation w.r.t. to the Lagrangian coordinates.

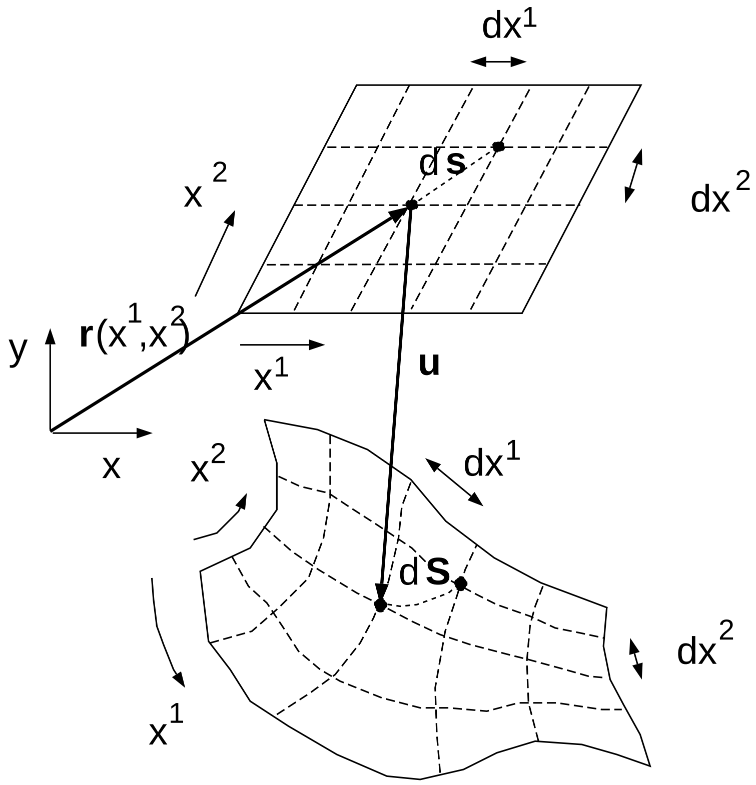

Figure 1: Sketch of the deformation of a (two-dimensional) body. The body-fitted Lagrangian coordinate system is deformed with the body. During the deformation, the infinitesimal line element $d\mathbf{s}$ is deformed to $d\mathbf{S}$.

3 The Deformation/Review of Differential Geometry

3.1 The co- and contravariant base vectors

- Use Lagrangian coordinates x^i to label material points in the undeformed body:

$$\mathbf{r} = \mathbf{r}(x^i) \tag{1}$$

- Tangent vectors to coordinate lines

$$\mathbf{g}_i = \mathbf{r}_{,i} \tag{2}$$

are the *covariant* base vectors. **Note:** Generally, these base vectors vary from point to point and are not unit vectors.

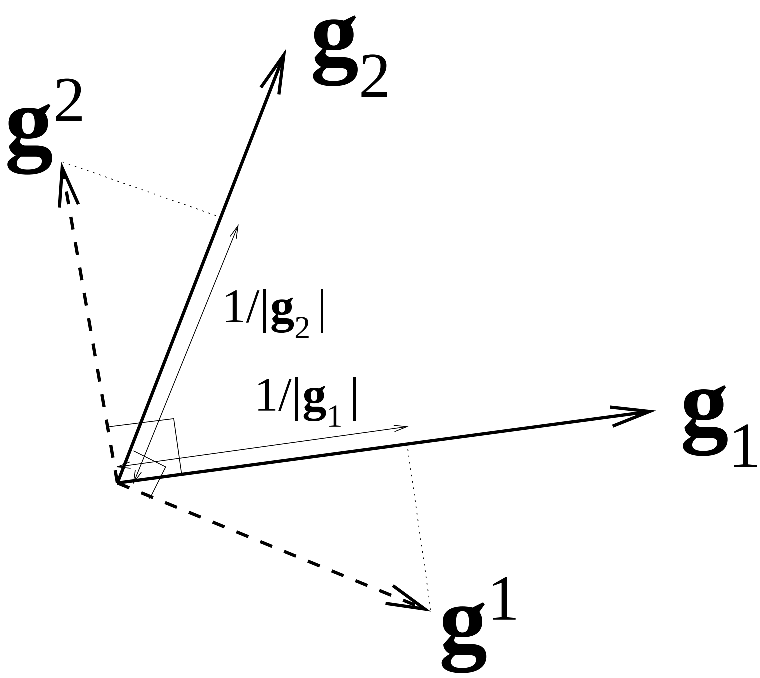

Figure 2: Sketch of the co- and contravariant base vectors.

- Define a second set of (*contravariant*) base vectors, $\mathbf{g}^j$, such that

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j \tag{3}$$

- Write the contravariant base vectors as a linear combination of the covariant base vectors:

$$\mathbf{g}^j = g^{ji}\mathbf{g}_i, \tag{4}$$

and vice versa

$$\mathbf{g}_i = g_{ij}\mathbf{g}^j. \tag{5}$$

The g^{ij} and g_{ij} are second order tensors, called the contravariant and covariant *metric tensors*, respectively.

- Dot multiplying the various basis vectors and using (3) shows that

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \quad \text{and} \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$$

and we have $g_{ij} = g_{ji}$ and $g^{ij} = g^{ji}$.

- Also:

$$g_{ij}g^{jk} = \delta_i^k$$

i.e. the g_{ij} and g^{ij} are each other's inverses (as matrices).

- Decompose vectors into either the covariant or the contravariant basis, e.g.

$$\mathbf{u} = u^i \mathbf{g}_i = u_i \mathbf{g}^i.$$

Using (5) in the first identity, we see that

$$\mathbf{u} = u^i \mathbf{g}_i = u^i g_{ij} \mathbf{g}^j = u_j \mathbf{g}^j.$$

Therefore, the co- and contravariant components of a vector $\mathbf{u}$, (u_i and u^i, respectively) are related by

$$u_j = g_{ij} u^i,$$

and similarly,

$$u^j = g^{ij} u_i.$$

In other words, the tensors g^{ji} and g_{ji} raise and lower the indices of the components of vectors (and higher order tensors).

- Apply to dot product:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u^i \mathbf{g}_i \cdot v^j \mathbf{g}_j \\ &= u^i v^j \; \mathbf{g}_i \cdot \mathbf{g}_j \\ &= u^i v^j g_{ij} \\ &= u^i v_i \end{aligned}$$

- Physical components:
  - Since the base vectors are not (necessarily) unit vectors, the components u^i of a vector $\mathbf{u} = u^i \mathbf{g}_i$ are not (necessarily) the 'lengths' of the vector in the directions of the respective base vectors. To obtain these (= the physical components):

$$\begin{aligned} \mathbf{u} &= u^i \mathbf{g}_i = u^{*i} \frac{\mathbf{g}_i}{|\mathbf{g}_{(i)}|} \\ &= u^{*i} \frac{\mathbf{g}_i}{\sqrt{g_{(ii)}}} \end{aligned}$$

and hence

$$u^{*i} = \sqrt{g_{(ii)}} u^i.$$

Similarly:

$$u_i^* = \sqrt{g^{(ii)}} u_i.$$

- Covariant derivatives:
  - Derivatives of vectors w.r.t. to Lagrangian coordinates:

$$\begin{aligned} \mathbf{u}_{,i} &= \left(u^j \mathbf{g}_j\right)_{,i} \\ &= u^j_{,i} \, \mathbf{g}_j + u^j \, \mathbf{g}_{j,i} \end{aligned}$$

  - Decompose derivative of vector into base vectors

$$\mathbf{u}_{,i} = u^j\Big|_i \mathbf{g}_j$$

where $u^j\big|_i$ is the covariant derivative

$$u^i\Big|_j = u^i_{,j} + \Gamma^i_{jk} u^k$$

and the

$$\Gamma^k_{ij} = \mathbf{g}_{i,j} \cdot \mathbf{g}^k$$

are the Christoffel symbols.

3.2 Line, area and volume elements

- An infinitesimal line element $d\mathbf{s}$ in the undeformed configuration is given by

$$d\mathbf{s} = \mathbf{r}_{,i}\, dx^i = \mathbf{g}_i\, dx^i$$

and the square of its length is

$$\begin{aligned} (ds)^2 &= \mathbf{g}_i dx^i \cdot \mathbf{g}_j dx^j \\ &= \mathbf{g}_i \cdot \mathbf{g}^k\, g_{jk}\, dx^i dx^j \\ &= \delta_i^k\, g_{jk}\, dx^i dx^j \\ (ds)^2 &= g_{ji}\, dx^i dx^j. \end{aligned} \tag{6}$$

This explains the expression 'metric tensor'.

- An infinitesimal area element in the plane $x^3 = const.$ in the undeformed body is given by

$$\begin{aligned} da_3 &= |\mathbf{r}_{,1}\, dx^1 \times \mathbf{r}_{,2}\, dx^2| \\ &= |\mathbf{g}_1 \times \mathbf{g}_2|\, dx^1 dx^2 \end{aligned}$$

Equation (3) shows that the cross product between $\mathbf{g}_1$ and $\mathbf{g}_2$ is parallel to $\mathbf{g}^3$, therefore we can write the last equation as

$$\begin{aligned} da_3 &= (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \frac{\mathbf{g}^3}{\sqrt{g^{33}}}\, dx^1 dx^2 \\ &= (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_j \frac{g^{3j}}{\sqrt{g^{33}}}\, dx^1 dx^2. \end{aligned}$$

Since $\mathbf{g}_1$ and $\mathbf{g}_2$ are both perpendicular to $\mathbf{g}_1 \times \mathbf{g}_2$, only the component $j = 3$ contributes to the sum in the last equation and we have

$$da_3 = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 \sqrt{g^{33}}\, dx^1 dx^2.$$

It can easily be shown that (exercise)

$$(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 = \sqrt{g},$$

where

$$g = det\ g_{ij},$$

therefore we have

$$da_3 = \sqrt{g^{33} g}\ dx^1 dx^2. \tag{7}$$

Similar equations hold for da_1 and da_2.

- The volume of an infinitesimal parallelepiped in the undeformed body is given by

$$\begin{aligned} dv &= (\mathbf{r}_{,1} \times \mathbf{r}_{,2}) \cdot \mathbf{r}_{,3}\ dx^1 dx^2 dx^3 \\ &= (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3\ dx^1 dx^2 dx^3 \\ &= \sqrt{g}\ dx^1 dx^2 dx^3. \end{aligned}$$

3.3 Gauss' theorem

- For vector field $\mathbf{u}$:

$$\int\int\int \text{div}\ \mathbf{u}\ dv = \int\int \mathbf{u} \cdot \mathbf{n}\ da$$

- For tensorial representation $\mathbf{u} = u^i \mathbf{g}_i$:

$$\int\int\int \frac{1}{\sqrt{g}} \left(u^i \sqrt{g}\right)_{,i} dv = \int\int\int \left(u^i \sqrt{g}\right)_{,i} dx^1 dx^2 dx^3 = \int\int u^i\ n_i\ da \tag{8}$$

where $\mathbf{n} = n_i \mathbf{g}^i$ is the unit outer normal on the surface a.

3.4 The Deformation and the Strain Tensor

- The displacement field $\mathbf{u}$ moves points which were at a position $\mathbf{r}(x^i)$ in the undeformed body to a new position

$$\mathbf{R}(x^i) = \mathbf{r}(x^i) + \mathbf{u}(x^i).$$

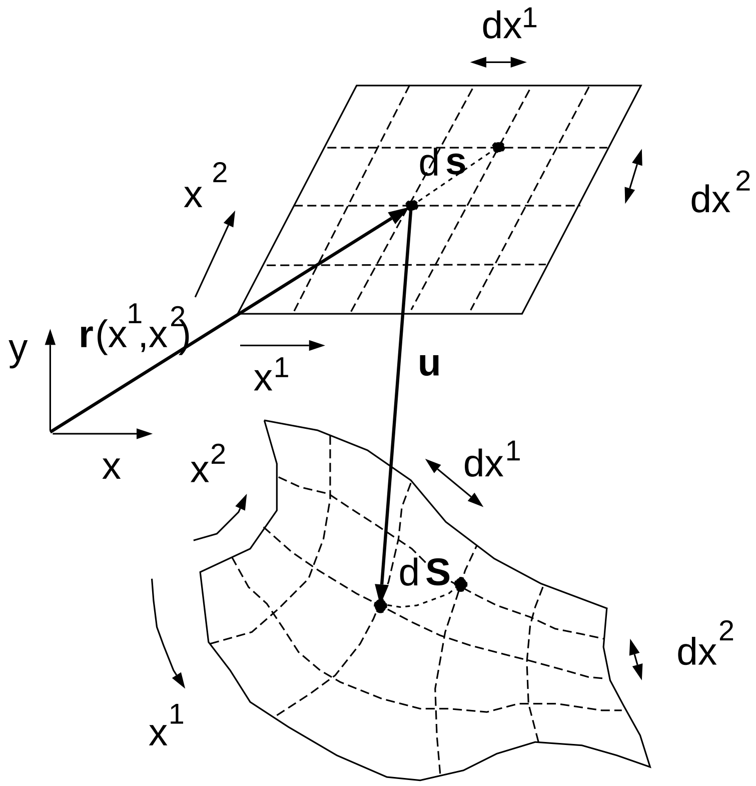

Figure 3: Sketch of the deformation of a (two-dimensional) body. The body-fitted Lagrangian coordinate system is deformed with the body. During the deformation, the infinitesimal line element $d\mathbf{s}$ is deformed to $d\mathbf{S}$.

- The deformed base vectors are given by

$$\mathbf{G}_i = \mathbf{R}_{,i} = \mathbf{r}_{,i} + \mathbf{u}_{,i},$$

i.e.

$$\mathbf{G}_i = \mathbf{g}_i + \mathbf{u}_{,i} \tag{9}$$

- The $\mathbf{G}_i$ are the tangent vectors to the body-attached coordinate lines $x^i = const.$ in the deformed configuration.

- The metric tensors of the body-attached coordinate system in the deformed body are given by

$$G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j = G_{ji} \quad \text{and} \quad G^{ij} = \mathbf{G}^i \cdot \mathbf{G}^j = G^{ji}. \qquad (10)$$

- The expressions for line elements $d\mathbf{S}$, area elements $d\mathcal{A}_i$ and volume elements dV in the deformed body are obtained from the ones for the undeformed body by replacing all lowercase variables with uppercase variables.

- The metric tensor is a measure of the lengths of infinitesimal line elements → use the difference between the deformed and undeformed metric tensors as a measure of the deformation and define

$$\epsilon_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right). \qquad (11)$$

  This tensor is known as *Green's Strain Tensor.*

- A physical interpretation of Green's Strain Tensor is given by the quadratic form

$$\epsilon_{ij}\, dx^i dx^j = \frac{1}{2}\left((dS)^2 - (ds)^2\right).$$

  → ϵ_{ij} characterizes the change of the *square* of infinitesimal line elements in the deformed and undeformed body.

- Green's strain tensor in terms of the displacements:

$$\epsilon_{ij} = \frac{1}{2}\left(\mathbf{g}_i \cdot \mathbf{u}_{,j} + \mathbf{g}_j \cdot \mathbf{u}_{,i} + \mathbf{u}_{,i} \cdot \mathbf{u}_{,j}\right). \qquad (12)$$

- Since Green's strain tensor is based on the change of lengths of line elements during the deformation, it is an *objective measure of the deformation*, i.e. its components are identically zero for rigid body displacements and rotations.

- For small displacements (and their derivatives) Green's strain tensor can be linearised to

$$\epsilon_{ij}^{(lin.)} = e_{ij} = \frac{1}{2}\left(\mathbf{g}_i \cdot \mathbf{u}_{,j} + \mathbf{g}_j \cdot \mathbf{u}_{,i}\right). \tag{13}$$

  where e_{ij} is the symmetric part of the displacement gradient tensor.

- Theories based on (12) and (13) are referred to as geometrically non-linear and geometrically linear theories, respectively. Geometrically non-linear theories have to be used for deformations in which the displacement gradients are large.

- Note that Green's strain tensor is objective while the linearised strain tensor is not.