Chapter 3

Vectors

3.1 Physical motivation

- Many physical quantities (e.g. temperature, density, pressure) are completely determined by the specification of their magnitude. We call such quantities *scalars*.
- Other physical quantities (e.g. velocity, force, acceleration) also have a direction. Such quantities are called *vectors*.

3.2 Definitions and conventions

- A vector **F** is completely determined by the specification of either

  - its magnitude and direction (see equation (3.3) below)

  or

  - its components parallel to the axes of a cartesian coordinate system (see Fig. 3.1).

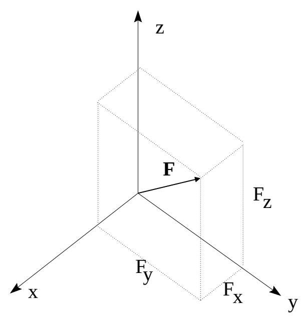

Figure 3.1: A three-dimensional vector **F** and its cartesian components (F_x, F_y, F_z).

- It is common practice to use boldface letters to represent vectors in order to distinguish them from scalars for which we use normal typeface. In handwriting, vectors are usually underlined. In some books, a little arrow is placed over the variable representing a vector. If we want to explicitly refer to a vector by its components we place them in brackets. Hence the following notations are equivalent

$$\mathbf{F} = \underline{F} = \vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = (F_x, F_y, F_z)\,.$$

- Note that the origin of the vector is irrelevant unless we are dealing with *position vectors* which point from the origin of the coordinate system to a certain point in space.

- Fig. 3.1 shows that the length (or the magnitude) of a vector is given by

$$|\mathbf{F}| = \left| \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \right| = \sqrt{F_x^2 + F_y^2 + F_z^2}.$$

3.3 Elementary vector operations

3.3.1 Vector addition

Two vectors are added by adding their respective components, i.e.

$$\mathbf{F} + \mathbf{G} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} + \begin{pmatrix} G_x \\ G_y \\ G_z \end{pmatrix} = \begin{pmatrix} F_x + G_x \\ F_y + G_y \\ F_z + G_z \end{pmatrix}. \tag{3.1}$$

Fig. 3.2 shows the graphical interpretation of equation (3.1).

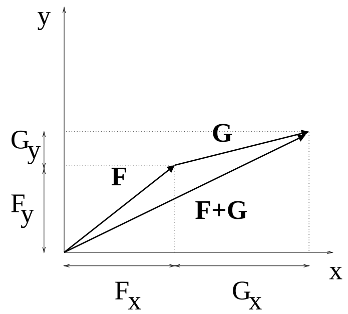

Figure 3.2: Illustration of vector addition in two dimensions.

3.3.2 Multiplication by a scalar

A vector is multiplied by a scalar by multiplying its components individually by that scalar, i.e.

$$\lambda\ \mathbf{F} = \lambda \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \lambda\ F_x \\ \lambda\ F_y \\ \lambda\ F_z \end{pmatrix}. \tag{3.2}$$

Fig. 3.3 shows that the multiplication by scalar changes the vector's length without changing its direction.

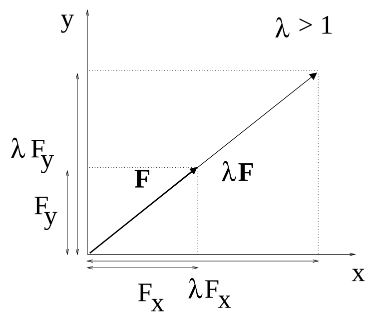

Figure 3.3: Illustration of the multiplication of a vector by a scalar in two dimensions (for $\lambda > 1$).

3.4 Unit vectors

- A special class of vectors is given by those vectors whose length is equal to one. We call such vectors *unit vectors.*
- Note that every vector $\mathbf{F}$ can be written as a product of its (scalar) length $|\mathbf{F}|$ with a unit vector $\mathbf{e_F}$ which points in the same direction, i.e.

$$\mathbf{F} = |\mathbf{F}|\ \mathbf{e_F}. \tag{3.3}$$

This expression specifies the vector in the 'magnitude and direction' form, mentioned in section 3.2.

- From equation (3.3) we obtain an expression for the unit vector $\mathbf{e_F}$ parallel to $\mathbf{F}$

$$\mathbf{e_F} = \frac{1}{|\mathbf{F}|}\ \mathbf{F}$$

- Three special unit vectors are given by the unit vectors parallel to the coordinate axes. We denote these vectors by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ (see Fig. 3.4). Using these unit vectors we have yet another representation of a general vector $\mathbf{F}$ in the form

$$\mathbf{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = F_x\ \mathbf{i} + F_y\ \mathbf{j} + F_z\ \mathbf{k}.$$

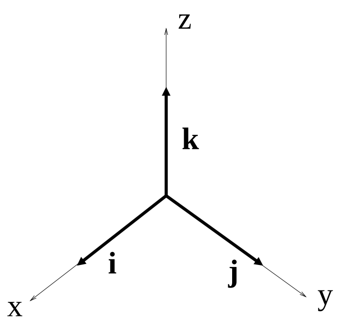

Figure 3.4: The unit vectors parallel to the coordinate axes, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

3.5 Vector products

- There are various ways in which we can form products of two vectors. They all have useful physical applications.

3.5.1 The scalar product

- As the name suggests, the results of the scalar multiplication of two vectors is a scalar. The scalar product[1] (indicated by a dot between the two vectors) is defined as

$$\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x b_x + a_y b_y + a_z b_z.$$

- An alternative representation of the scalar product is given by

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos\alpha$$

where α is the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$.

[1] The scalar product is also known as the inner product or the dot product.

- The latter equation provides a useful formula for the determination of the angle α between two vectors
$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{|\mathbf{a}|\ |\mathbf{b}|}$$
which is readily evaluated if the components of $\mathbf{a}$ and $\mathbf{b}$ are given.

- A physical application for the scalar product can be found in the statement "work = force times displacement" which clearly constitutes a product between two vectors (the force $\mathbf{F}$ and the displacement $\mathbf{s}$) and yields a scalar quantity (work w). Hence the appropriate product is the scalar product and we can write
$$w = \mathbf{F}\cdot\mathbf{s} = |\mathbf{F}||\mathbf{s}|\cos\alpha.$$
Here α is the angle between the force and the displacement and the $\cos\alpha$ term reflects the fact that only the component of the force which is parallel to the displacement, $|\mathbf{F}|\cos\alpha$, does work (see Fig. 3.5).

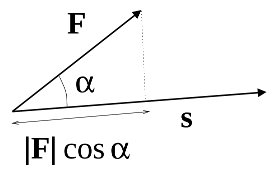

Figure 3.5: Work is the scalar product between the force, $\mathbf{F}$, and the displacement, $\mathbf{s}$: Only the component of the force which is parallel to the displacement does work.

3.5.2 The vector product

- As the name suggests, the results of the vector multiplication of two vectors $\mathbf{a}$ and $\mathbf{b}$ is another vector. We denote the vector product[2] by a cross between the two vectors, i.e. $\mathbf{a}\times\mathbf{b}$. In many books the vector product is represented by a wedge symbol, i.e. $\mathbf{a}\wedge\mathbf{b}$.

- Let $\mathbf{n}$ be a *unit vector* normal to $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}$ and $\mathbf{n}$ form a right handed system (see Fig. 3.6). Then
$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \sin\alpha\ \mathbf{n},$$
where α is the angle between $\mathbf{a}$ and $\mathbf{b}$.

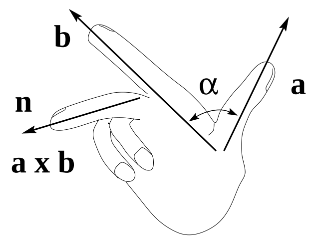

Figure 3.6: The right hand rule for vector products: The vector $\mathbf{a}\times\mathbf{b}$ points in the direction normal to both $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ form a right handed system.

- Similarly to the scalar product, the vector product can be expressed in terms of the components of $\mathbf{a}$ and $\mathbf{b}$:
$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}.$$

- The cross product between two vectors can also be written as a determinant involving the three cartesian basis vectors $\mathbf{i}$,$\mathbf{j}$ and $\mathbf{k}$ and the components of the two vectors. A convenient way to evaluate this determinant is illustrated in Fig. 3.7

[2]The vector product is also known as the outer product or the cross product.

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \text{x} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix} = (a_y b_z - a_z b_y)\,\mathbf{i} + (a_z b_x - a_x b_z)\,\mathbf{j} + (a_x b_y - a_y b_x)\,\mathbf{k}$$

Evaluation of the determinant via:

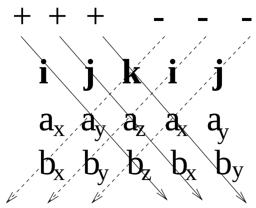

Figure 3.7: Evaluation of the vector product via a determinant.

- A geometrical interpretation of the vector product is given in Fig. 3.8: $|\mathbf{a}|\ |\mathbf{b}|\ \sin\alpha$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

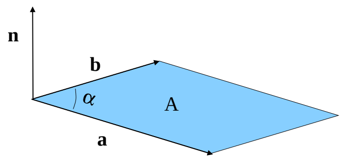

Figure 3.8: Geometrical interpretation of the vector product $\mathbf{a} \times \mathbf{b} = A\ \mathbf{n}$: The magnitude of $\mathbf{a} \times \mathbf{b}$ is given by the area A of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. The vector $\mathbf{a} \times \mathbf{b}$ is normal to $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ form a right handed system.

- Note that we cannot change the order of the vectors in the vector product since $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$. In fact,
$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$
- The vector product of two parallel vectors is zero, i.e.
$$\mathbf{a} \times \mathbf{b} = \mathbf{0} \quad \text{for} \quad \mathbf{a} \mid\mid \mathbf{b}\ .$$
- Vector products of the sums of vectors can simply be multiplied out while keeping the order of the vectors, e.g.
$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{d}.$$
- Scalar factors can be extracted from the vector products, i.e.
$$(\lambda\mathbf{a}) \times (\mu\mathbf{b}) = \lambda\mu\ (\mathbf{a} \times \mathbf{b}).$$
- A physical application for the vector product is sketched in Fig. 3.9: The moment $\mathbf{M}$ that a force $\mathbf{F}$, acting at a point Q in a body, exerts about another point P is given by
$$\mathbf{M} = \mathbf{r} \times \mathbf{F},$$
where $\mathbf{r}$ is the vector from P to Q.

  The vectorial moment is related to the sense of rotation by yet another right hand rule as illustrated in Fig. 3.10

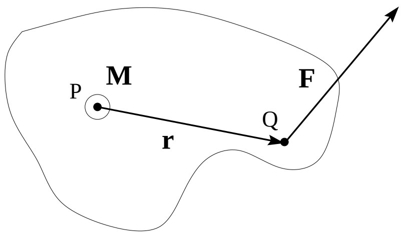

Figure 3.9: Physical application of the vector product: The force $\mathbf{F}$, acting at point Q, exerts a moment $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ about the point P. If $\mathbf{r}$ and $\mathbf{F}$ lie in the plane of the paper, the moment vector $\mathbf{M}$ points vertically out of that plane.

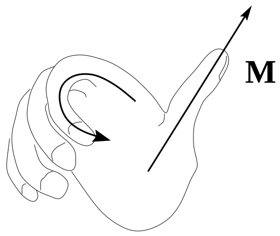

Figure 3.10: Sketch of the relation between the sense of rotation associated with the moment vector $\mathbf{M}$.

3.5.3 The triple product

- The triple product is a product of three vectors, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, and yields a scalar V via the following operation
$$V = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$
- Geometrically, V represents the volume of the cuboid (also called parallelepiped) spanned by the three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, as illustrated in Fig. 3.11. The triple product is positive (negative) if $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ form a right handed (left handed) system.

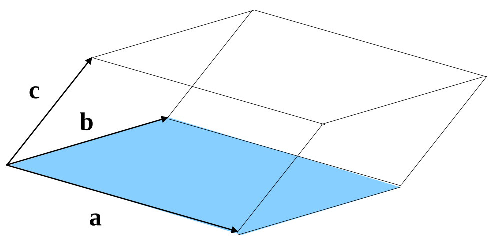

Figure 3.11: The triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ represents the volume of the cuboid spanned by the three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

3.6 The vector equation of lines in space

- A line in space is completely determined by the position vector to one point ($\mathbf{a}$, say) on the line **and either**

  - the position vector to a second different point ($\mathbf{b}$, say) on that line,

  or

  - the tangent vector $\mathbf{t}$ to that line.

- The position vector $\mathbf{l}$ to an arbitary point on that line is then given by either

$$\mathbf{l} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}), \tag{3.4}$$

or

$$\mathbf{l} = \mathbf{a} + \lambda\mathbf{t}. \tag{3.5}$$

- In these two representations, varying the scalar $\lambda \in [-\infty, \infty]$ moves the point $\mathbf{l}$ along the line; see Fig. 3.12.

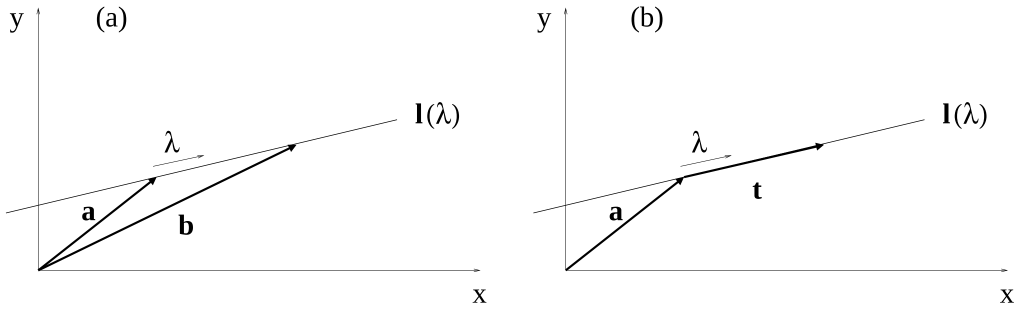

Figure 3.12: Two-dimensional illustration of the two ways in which a line in space can be defined.

- Equations (3.4) and (3.5) are equivalent since $(\mathbf{b} - \mathbf{a})$ is a tangent vector to the straight line $\mathbf{l}$.
- If $|\mathbf{b} - \mathbf{a}| = 1$ or if $|\mathbf{t}| = 1$ then λ measures the distance of $\mathbf{l}$ from $\mathbf{a}$ and represents the arclength along the line.

3.6.1 The intersection of lines in space

- Geometrical considerations show that, depending on their relative positions and directions, two lines in space can have

  (i) no intersection

  (ii) one intersection

  (iii) infinitely many intersections – this somewhat trival case corresponds to the situation in which the two lines are, in fact, identical.

- Assume the vector equations of two lines are given in the form

$$\mathbf{l}_1(\lambda_1) = \mathbf{a}_1 + \lambda_1\mathbf{t}_1 \tag{3.6}$$

and

$$\mathbf{l}_2(\lambda_2) = \mathbf{a}_2 + \lambda_2\mathbf{t}_2. \tag{3.7}$$

Note that both lines have their own parameter λ_1 and λ_2, respectively.

- The condition for the lines to intersect is

$$\mathbf{l}_1(\lambda_1) = \mathbf{l}_2(\lambda_2). \tag{3.8}$$

- This represents three equations (one for each component of this vector equation) for the two unknowns λ_1 and λ_2. In general, this is an overdetermined system of equations for which solutions exist only in special circumstances.
- The number of solutions to (3.8) corresponds to the number of intersections that the two lines have: none, one or infinitely many. Which situation exist becomes clear when one tries to solve (3.8). If there is a contradiction between the equations then there is no solution; if the equations can be solved uniquely then there is exactly one solution; if all three equations are linearly dependent then there are infinitely many solutions.
- If only one intersection exists, substitution of the unique value of λ_1 into (3.6) (or λ_2 into (3.7)) provides the position at which the lines intersect.

3.7 The vector equation of planes in space

- A plane in space is completely determined by the position vector to one point ($\mathbf{a}$, say) on the line **and either**
  - the position vectors to two different points ($\mathbf{b}$ and $\mathbf{c}$, say) in that plane such that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ do not lie in a straight line,

  or

  - two non-parallel tangent vectors ($\mathbf{t}_1$ and $\mathbf{t}_2$, say) to that plane.

  or

  - a suitable combination of the two previous options.
- The position vector $\mathbf{p}$ to an arbitary point on that plane is then given by either

$$\mathbf{p} = \mathbf{a} + \lambda_1(\mathbf{b} - \mathbf{a}) + \lambda_2(\mathbf{c} - \mathbf{a}), \tag{3.9}$$

or

$$\mathbf{p} = \mathbf{a} + \lambda_1\mathbf{t}_1 + \lambda_2\mathbf{t}_2. \tag{3.10}$$

- In these two representations, varying the scalars $\lambda_1, \lambda_2 \in [-\infty, \infty]$ moves the point $\mathbf{p}$ around the entire plane; see Fig. 3.13.

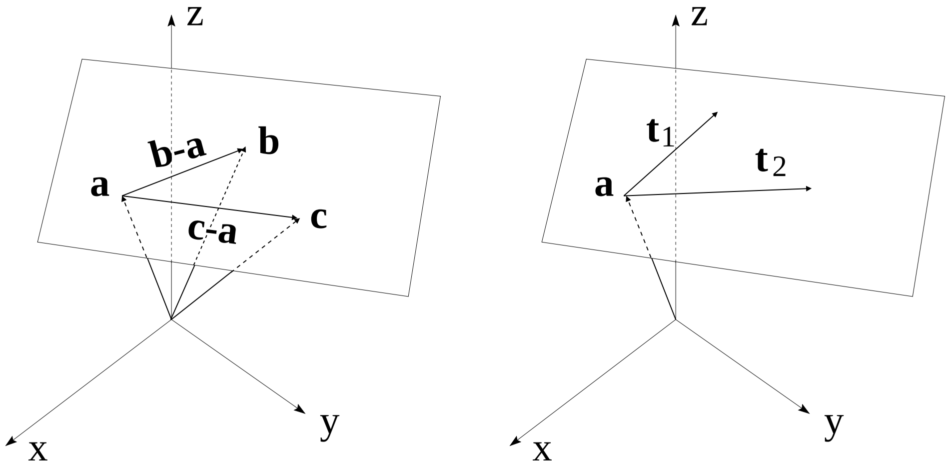

Figure 3.13: Illustration of the two ways in which a plane in space can be defined.

- The two representations of the plane are equivalent since under the above conditions the two vectors $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ are two non-parallel tangent vectors to the plane.

3.7.1 The intersection of two planes

- Geometrical considerations show that, depending on their relative positions and directions, two planes can either
  - **(i)** have no intersection (in this case they are parallel but separated by a finite distance)
  - **(ii)** intersect each other along a single line in space
  - **(iii)** have infinitely many intersections (this trival case corresponds to the situation in which the two planes are, in fact, identical).

- The condition for intersection of the two planes given by

$$\mathbf{p}_1(\lambda_1, \lambda_2) = \mathbf{a} + \lambda_1 \mathbf{t}_1 + \lambda_2 \mathbf{t}_2$$

and

$$\mathbf{p}_2(\mu_1, \mu_2) = \mathbf{b} + \mu_1 \mathbf{s}_1 + \mu_2 \mathbf{s}_2$$

is

$$\mathbf{p}_1(\lambda_1, \lambda_2) = \mathbf{p}_2(\mu_1, \mu_2).$$

- This equation has to be fulfilled in all three components, therefore we have three equations for the four unknowns $\lambda_1, \lambda_2, \mu_1$ and μ_2. The solvability of this underdetermined system of three equations reflects the three geometrical cases mentioned above.

- In 'most cases'[3], the solution of this system will have a free parameter which (geometrically speaking) parametrises the line of intersection. In special cases (corresponding to the cases (i) and (iii) above) the system has either no solution or has a solution for every combination of λ_1 and λ_2.

3.7.2 The normal on a plane

- A normal vector $\mathbf{N}$ on a plane in space can easily be constructed by forming the cross product of the two tangent vectors to the plane, i.e.

$$\mathbf{N} = \mathbf{t}_1 \times \mathbf{t}_2$$

or, if a unit normal $\mathbf{n}$ is required, by normalising this vector via

$$\mathbf{n} = \frac{\mathbf{t}_1 \times \mathbf{t}_2}{|\mathbf{t}_1 \times \mathbf{t}_2|}.$$

3.7.3 The angle between two planes

- The angle between two intersecting planes is given by

$$\cos\alpha = |\mathbf{n}_1 \cdot \mathbf{n}_2|$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit normal vectors on the two planes and α is taken to lie in the range between 0 and $\pi/2$.

3.7.4 The distance of a point from a plane

- The distance d of a point (whose position vector is $\mathbf{q}$) from a plane given in the form (3.9) or (3.10) is

$$d = |\mathbf{n} \cdot (\mathbf{q} - \mathbf{a})|$$

where $\mathbf{n}$ is the unit normal on the plane.

[3] I apologise for this slightly unmathematical statement...