Chapter 2

Second order linear ODEs with constant coefficients

2.1 Definition

- The generic second order linear ordinary differential equation (ODE) with constant coefficients has the form
$$m\ddot{x} + k\dot{x} + cx = F(t), \tag{2.1}$$
where m, k and c are constants and $F(t)$ is a given function of the independent variable t. The dot indicates differentiation of the unknown function $x(t)$ with respect to t, i.e. $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$.

- For the purpose of this lecture we will restrict ourselves to the case where the constants m, k and c are real.

- Equation (2.1) is of *second order* because it contains up to second derivatives of the unknown function $x(t)$; it is *linear* in the unknown function $x(t)$; it is an *ordinary differential equation* because $x(t)$ only depends on one independent variable t and it has *constant coefficients* as opposed to coefficients which vary with t.

- In order to find a unique solution to equation (2.1), we need to specify two *initial conditions* which prescribe $x(t)$ and its first derivative $\dot{x}(t)$ at some given value of t (e.g. $t = 0$) i.e.
$$x(t) = x_0 \quad \text{and} \quad \dot{x}(t) = v_0 \quad \text{for } t = 0, \tag{2.2}$$
where x_0 and v_0 are given constants.

2.2 Physical motivation

- Second order linear ODEs with constant coefficients appear in many physical applications. One physical systems whose behaviour is governed by an ODE of the form given in (2.1) is the linear mechanical oscillator shown in Fig. 2.1. In this case, m represents the mass of the particle attached to the spring, k is a measure of the strength of the damper and c represents the spring stiffness. $F(t)$ is the applied external force. t represents time and $x(t)$ is the displacement of the mass from its rest position.

- The physical interpretation of the individual terms in equation (2.1) is as follows. The term $k\dot{x}$ represents the force exerted by the damper on the mass: for a linear damper, this force is proportional to the velocity $\dot{x}$ and it resists the motion. The term cx represents the force exerted by the spring on the mass: for a linear spring, this force is proportional to the displacement x and it acts in the direction opposite to the displacement. Equation (2.1) expresses Newton's law, which states that the sum of all forces acting on the particle is equal to its mass times its acceleration, $m\ddot{x}$.

- The initial conditions represent the initial displacement, x_0, and the initial velocity v_0 of the mass at time $t = 0$.

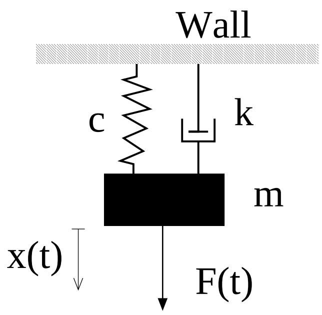

Figure 2.1: Sketch of a linear mechanical oscillator consisting of a mass m attached to a fixed wall by a linear spring of spring stiffness c and a damper of damping constant k. A time-dependent force $F(t)$ is applied to the mass and the displacement of the mass from its rest position is represented by $x(t)$.

2.3 The homogeneous equation

- If $F(t) = 0$, equation (2.1) is reduced to its *homogeneous* form,

$$m\ddot{x} + k\dot{x} + cx = 0. \tag{2.3}$$

- The homogeneous equation (2.3) can be solved by looking for a solution in the form $x(t) = Ae^{\lambda t}$, where A and λ are constants to be determined such that $x(t)$ fulfills (2.3) and the initial conditions (2.2).
- λ can be determined by the following procedure:
  - Insert $x(t) = Ae^{\lambda t}$ into equation (2.3).
  - Carry out the differentiations and cancel the common factor $Ae^{\lambda t}$.
  - This transforms the ODE into the *characteristic polynomial*

  $$m\lambda^2 + k\lambda + c = 0, \tag{2.4}$$

  which is a quadratic equation in the unknown λ.
  - The characteristic polynomial has two (not necessarily distinct!) roots which we call λ_1 and λ_2. Note that in general λ_1 and λ_2 are complex.
- The functions $x_1(t) = Ae^{\lambda_1 t}$ and $x_2(t) = Be^{\lambda_2 t}$ are solutions of (2.3) for *any* value of the arbitrary constants A and B.
- The form of the *general solution* to (2.3) depends on the values λ_1 and λ_2. We distinguish four cases

Case I: λ_1 and λ_2 are real and distinct

In this case the general solution is given by

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \tag{2.5}$$

for *any* value of the two constants A and B.

Case II: $\lambda_1 = \lambda_2 = \lambda$

If this occurs then λ can be shown to be real and the general solution is given by

$$x(t) = Ae^{\lambda t} + Bte^{\lambda t} \tag{2.6}$$

for *any* value of the two constants A and B. Note that the second exponential is multiplied by t and thus presents a second independent solution.

Case III: λ_1 and λ_2 are complex

If this occurs then λ_1 and λ_2 can be shown to be complex conjugates, i.e. they can be written in the form

$$\lambda_1 = a + ib \quad \text{and} \quad \lambda_2 = a - ib. \tag{2.7}$$

In this case the general solution is again given by

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \tag{2.8}$$

for *any* value of the two constants A and B. The special form of λ_1 and λ_2 and the definition of the complex exponential (see (1.21)) enables us to to rewrite the solution in the form

$$x(t) = e^{at}\left(C\cos(bt) + D\sin(bt)\right) \tag{2.9}$$

for (different) arbitrary constants C and D.

Case IV: λ_1 and λ_2 are purely imaginary

This is Case III with $a = 0$. λ_1 and λ_2 are still complex conjugates and thus have the form

$$\lambda_1 = ib \quad \text{and} \quad \lambda_2 = -ib. \tag{2.10}$$

Hence, the general solution is given by

$$x(t) = Ae^{ibt} + Be^{-ibt} \tag{2.11}$$

for *any* value of the two constants A and B. Using Euler's formula, we can rewrite this as

$$x(t) = C\cos(bt) + D\sin(bt) \tag{2.12}$$

for (different) arbitrary constants C and D.

2.4 Initial Conditions

- The previous section showed that the general solution of the homogeneous equation has two free constants A and B. They need to be chosen such that the initial conditions are fulfilled.
- This is done by inserting the general solution into the two initial conditions (2.2). This provides two equations from which the two unknowns A and B can be determined.

2.5 Eigenfrequencies

- More insight into the behaviour of the solutions to (2.3) can be gained by rewriting the equation in the following form

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = 0, \tag{2.13}$$

where

$$\delta = \frac{k}{2m} \quad \text{and} \quad \omega^2 = \frac{c}{m}. \tag{2.14}$$

- Note that in a physical system, such as the mechanical oscillator sketched in Fig. 2.1, the coefficients m, k and c are positive. Hence δ and ω^2 are positive as well.
- The solution to the corresponding characteristic equation

$$\lambda^2 + 2\delta\lambda + \omega^2 = 0, \tag{2.15}$$

is given by

$$\lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega^2} \tag{2.16}$$

or, written in a different form,

$$\lambda_{1,2} = -\delta \pm i\sqrt{\omega^2 - \delta^2}. \tag{2.17}$$

- This allows a straightforward identification of the four cases, discussed in section 2.3:

Case I: $\delta > \omega$

In this case (2.16) shows that both roots are real and the solution is given by

$$x(t) = Ae^{(-\delta+\sqrt{\delta^2-\omega^2})t} + Be^{(-\delta-\sqrt{\delta^2-\omega^2})t}. \tag{2.18}$$

Since both exponents are negative, this represents a *purely damped motion*, i.e. the system does not perform any oscillations.

Case II: $\delta = \omega$

In this case the square root in (2.16) vanishes and we have $\lambda_1 = \lambda_2 = -\delta$ and the general solution is given by

$$x(t) = Ae^{-\delta t} + Bte^{-\delta t}. \tag{2.19}$$

This represents a *critically damped motion* in which $x(t)$ approaches zero but can cross the value $x = 0$ (at most) once (when $t = -A/B$; if this happens for $t > 0$ depends on the initial conditions which determine A and B).

Case III: $\delta < \omega$

In this case (2.17) shows that both roots are complex. The general solution is then given by

$$x(t) = e^{-\delta t}\left(A\cos(t\sqrt{\omega^2-\delta^2}) + B\sin(t\sqrt{\omega^2-\delta^2})\right). \tag{2.20}$$

This solution represents a *damped oscillation* with frequency $\sqrt{\omega^2-\delta^2}$ whose amplitude decays exponentially.

Case IV: $\delta = 0$

The solution (2.20) is still valid and for $\delta = 0$ we obtain

$$x(t) = A\cos(\omega t) + B\sin(\omega t), \tag{2.21}$$

which is an *undamped oscillatory motion* with eigenfrequency ω.

2.6 The inhomogeneous equation/forced motions

- The general solution of the inhomogeneous equation

  $$\ddot{x} + 2\delta\dot{x} + \omega^2 x = F(t), \tag{2.22}$$

  is given by

  $$x(t) = x_h(t) + x_p(t). \tag{2.23}$$

- In (2.23), $x_h(t)$ is the general solution of the corresponding homogeneous equation (2.13) and thus contains the two free constants required to fulfill the initial conditions. $x_p(t)$ (the particular solution) is *any* solution of the inhomogeneous equation.

- The solution $x_h(t)$ of the homogeneous equation is often called the *complementary function.*

- Note that we still have to find one (but only one!) particular solution to the inhomogeneous equation. In general, this is difficult but for certain forms of the forcing function $F(t)$, particular solutions $x_p(t)$ can be found systematically.

2.6.1 Polynomial forcing

- If $F(t)$ is an N-th order polynomial in t, i.e. if it has the form

$$F(t) = f_0 + f_1 t + f_2 t^2 + ... + f_N t^N = \sum_{n=0}^{N} f_n t^n$$

with given coefficients $f_0, ..., f_N$, then a particular solution can be found by the following procedure:

1. Look for a solution in the form
$$x_p(t) = X_0 + X_1 t + X_2 t^2 + ... + X_N t^N = \sum_{n=0}^{N} X_n t^n. \tag{2.24}$$
Note that you will have to include *all* powers of t up to t^N, even if some powers of t are missing in the polynomial for $F(t)$.
2. Insert (2.24) into the inhomogeneous ODE (2.22) and carry out the differentiations.
3. Collect all terms that are multiplied by the same power of t.
4. Set the terms multiplying each power of t to zero. This provides $N+1$ equations for the $N+1$ unknown coefficients $X_0, ..., X_N$.
5. Solve these equations for the $X_0, ..., X_N$ and insert them into (2.24).

2.6.2 Exponential forcing

- If $F(t)$ consists of a sum of exponentials, i.e. if it has the form

$$F(t) = \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t} + ... + \alpha_N e^{\beta_N t} = \sum_{n=1}^{N} \alpha_n e^{\beta_n t}.$$

with given coefficients $\alpha_0, ..., \alpha_N$ and $\beta_0, ..., \beta_N$ then a particular solution can be found by the following procedure:

1. Look for a solution in the form
$$x_p(t) = \gamma_1 e^{\beta_1 t} + \gamma_2 e^{\beta_2 t} + ... + \gamma_N e^{\beta_N t} = \sum_{n=1}^{N} \gamma_n e^{\beta_n t}. \tag{2.25}$$
Note that you have to look for a solution that contains the *same* exponents $\beta_1, ..., \beta_N$ as the forcing, but has different amplitudes $\gamma_1, ..., \gamma_N$.
2. Insert (2.25) into the inhomogeneous ODE (2.22) and carry out the differentiations.
3. Collect all terms that are multiplied by the same exponentials. Note that the terms don't 'mix' – you can, in fact, treat every exponential separately.
4. Set the terms multiplying each exponential to zero. This provides N equations for the N unknown coefficients $\gamma_1, ..., \gamma_N$.
5. Solve these equations for the $\gamma_1, ..., \gamma_N$ and insert them into (2.25).

2.6.3 Harmonic forcing

- If $F(t)$ has the form
$$F(t) = F\sin(\Omega t) \quad \text{or} \quad F(t) = F\cos(\Omega t)$$
it is advantageous to carry out the calculation with complex numbers by writing
$$F(t) = F e^{i\Omega t} = F\cos(\Omega t) + iF\sin(\Omega t). \tag{2.26}$$

Once the calculation is completed we can extract the relevant real solution (corresponding to the cos or sin forcing) by taking the real or imaginary parts of the complex solution.

- We know from the previous section that we can find a solution for an exponential forcing in the form
$$x_p(t) = Xe^{i\Omega t}. \tag{2.27}$$
Hence we insert (2.27) and (2.26) into (2.22) and carry out the differentiations.

- After cancelling the common factor $e^{i\Omega t}$, this provides the following equation for the unknown amplitude X (which is complex in general):
$$X = \frac{F}{(\omega^2 - \Omega^2) + i\ (2\delta\Omega)} = F\ \frac{(\omega^2 - \Omega^2) - i\ (2\delta\Omega)}{(\omega^2 - \Omega^2)^2 + (2\delta\Omega)^2} \tag{2.28}$$

- Now we can multiply out $x_p(t) = Xe^{i\Omega t} = X(\cos(\Omega t) + i\sin(\Omega t))$ and extract the real or imaginary part to obtain the relevant real solution, i.e.
$$x_p(t) = \mathcal{R}(X)\cos(\Omega t) - \mathcal{I}(X)\sin(\Omega t) \qquad \text{for} \quad F(t) = F\cos(\Omega t)$$
and
$$x_p(t) = \mathcal{R}(X)\sin(\Omega t) + \mathcal{I}(X)\cos(\Omega t) \qquad \text{for} \quad F(t) = F\sin(\Omega t).$$

- Alternatively, we can re-write (2.28) in polar form and thus obtain the *amplitude* of the response
$$|X| = \frac{F}{\sqrt{(\omega^2 - \Omega^2)^2 + (2\delta\Omega)^2}}$$
and the *phase angle*
$$\varphi = \arg(X) = \arctan\left(\frac{-2\delta\Omega}{\omega^2 - \Omega^2}\right). \tag{2.29}$$

- Hence, the particular solution can also be written as
$$x_p(t) = |X|\ e^{i(\Omega t + \varphi)}.$$
Comparing this to (2.26) shows that the particular solution (i.e. the forced motion), $x_p(t)$, has a phase difference of φ against the forcing, $F(t)$.

- Note that for weak damping (small δ), the amplitude of the response, $|X|$, becomes very large when the excitation frequency Ω is close to the eigenfrequency of the system, $\Omega \approx \omega$. This is known as *resonance*. For vanishing damping, $\delta = 0$, the response becomes unbounded when $\Omega \to \omega$. This is the 'resonance catastrophe'.

  In practical applications there is always some damping but the amplitudes of the forced oscillations of many physical systems can still become so large that the system is destroyed when the excitation frequency is sufficiently close to the eigenfrequency of the system.

- Note that equation (2.29) shows that resonance is accompanied by a 180^o jump in the phase angle.

- Finally, note that for positive damping ($\delta > 0$), the homogeneous solution $x_h(t)$ decays rapidly with increasing time (see section 2.5), i.e. $x_h(t) \to 0$. For sufficiently large times, only the forced motion $x_p(t)$ persists. Therefore, the solution $x_h(t)$ is often referred to as a *transient solution*.