MT1612 Lecture Notes

This set of notes summarises the main results of the lecture MT1612 (Mathematics for Chemistry and Material Science Students). Please email any corrections (yes, there might be the odd typo...) or suggestions for improvement to *M.Heil@maths.man.ac.uk* or see me after the lecture or in my office (Room 18.07 in the Mathematics Building).

Generally, the notes will be handed out after the material has been covered in the lecture. You can also download them from the WWW:

http://www.maths.man.ac.uk/˜mheil/Lectures/MT1612/MT1612.html.

This WWW page will also contain announcements, example sheets, solutions, etc.

Chapter 1

Complex Numbers

1.1 Definition

- Complex numbers are an extension of the real numbers. They arise naturally in many applications such as the solution of quadratic equations.
- A complex number z has the form
$$z = a + ib \tag{1.1}$$
where a and b are real numbers which represent the *real* and *imaginary* parts of z, respectively.
- i is the *imaginary unit* and has the property
$$i^2 = -1 \quad \text{i.e.} \quad i = \sqrt{-1}. \tag{1.2}$$
- The real and imaginary parts of a complex number z are often denoted by $Re(z), \mathcal{R}(z)$ and $Im(z), \mathcal{I}(z)$, respectively.

1.2 Fundamental operations with complex numbers

Basically, all operations with complex numbers are carried out using the same rules that apply to real numbers, replacing i^2 by -1 whenever it appears. Therefore, we obtain the following rules[1]

- **Addition:**
$$(a + ib)(c + id) = (a + c) + i(b + d) \tag{1.3}$$
- **Subtraction:**
$$(a + ib) - (c + id) = (a - c) + i(b - d) \tag{1.4}$$
- **Multiplication:**
$$(a + ib)(c + id) = (ac - bd) + i(ad + bc) \tag{1.5}$$
- **Division:**
$$\frac{(a + ib)}{(c + id)} = \frac{(a + ib)}{(c + id)}\frac{(c - id)}{(c - id)} = \frac{ac + bd}{c^2 + d^2} + i\,\frac{bc - ad}{c^2 + d^2} \tag{1.6}$$

Further operations with complex numbers include:

- **Complex Conjugate:** The complex conjugate z^* of a complex number $z = a + ib$ is given by $z^* = a - ib$. In many books the complex conjugate is denoted by $\bar{z}$.
- **Absolute Value/Modulus:** The absolute value/modulus of a complex number $z = a + ib$ is given by
$$|z| = |a + ib| = \sqrt{a^2 + b^2}. \tag{1.7}$$

[1] ...which you should *not* learn by heart!

1.3 Graphical and polar representation of complex numbers

- We can represent complex numbers graphically by plotting their real and imaginary parts along the x and y-axes of a two-dimensional Cartesian coordinate system (the 'complex plane') as illustrated in Fig. 1.1. This representation is known as the *Argand diagram.*

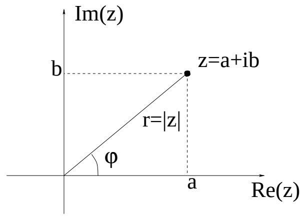

Figure 1.1: Graphical representation of a complex number $z = a + ib$ in the complex plane (Argand diagram).

- Fig. 1.1 suggests an alternative representation of complex numbers in polar form: The radial distance r of the complex number z from the origin of the complex plane is given by its modulus

$$r = |z| = \sqrt{a^2 + b^2}, \tag{1.8}$$

and the angle against the positive real axis, φ, is given by the *argument*[2]

$$\varphi = \arg z = \arctan(b/a). \tag{1.9}$$

- Using the modulus $r = |z|$ and argument $\varphi = \arg z$ we thus write complex numbers in *polar form* as

$$z = r(\cos\varphi + i\sin\varphi). \tag{1.10}$$

1.4 De Moivre's theorem

- The polar representation of complex numbers allows the derivation of simpler formulae for the multiplication and division of complex numbers: Two complex numbers $z_1 = x_1 + iy_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = x_2 + iy_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$ are multiplied by

$$z_1 z_2 = r_1 r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)) \tag{1.11}$$

and divided by

$$z_1/z_2 = r_1/r_2(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)) \tag{1.12}$$

or in words:

> Two complex numbers are multiplied (divided) by multiplying (dividing) their moduli and by adding (subtracting) their arguments.

- The following two formulae are a direct consequence of (1.11) and (1.12):

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{1.13}$$

and

$$|z_1 z_2| = |z_1||z_2|. \tag{1.14}$$

[2]**Note:** Don't blindly evaluate this expression with a pocket calculator! Always begin by sketching the position of the complex number in the Argand diagram to assess in what quadrant the complex number is located.

- Repeated application of (1.11) shows that the product of n different complex numbers is given by

$$z_1 z_2 ... z_n = r_1 r_2 ... r_n(\cos(\varphi_1 + \varphi_2 + ... + \varphi_n) + i\sin(\varphi_1 + \varphi_2 + ... + \varphi_n)). \tag{1.15}$$

- Setting all z_i in this product to the same value we obtain *De Moivre's theorem*:

$$z^n = [r(\cos(\varphi) + i\sin(\varphi))]^n = r^n(\cos(n\varphi) + i\sin(n\varphi)), \tag{1.16}$$

or in words:

> A complex number is raised to the n-th power by raising its modulus to the n-th power and multiplying its argument by n (see Fig. 1.2)

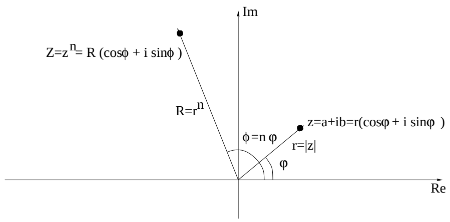

Figure 1.2: Graphical representation of De Moivre's theorem: Raising a complex number $z = r(\cos(\varphi) + i\sin(\varphi))$ to the n-th power changes its modulus, r, to r^n and its argument, φ, to $n\varphi$.

- De Moivre's theorem implies that

$$\arg(z^n) = n\arg(z) \quad \text{and} \quad |z^n| = |z|^n. \tag{1.17}$$

- Note that the power of a complex number can have an argument greater than 2π. Due to the periodicity of the trigonometric functions we can add and subtract multiples of 2π from the argument of any complex number without changing its value. We use the term *principal argument* for the argument of the complex number if it lies in the range between 0 and 2π.

1.5 Application of De Moivre's Theorem: Formulae for $\sin(n\varphi)$ and $\cos(n\varphi)$

De Moivre's Theorem can be used to transform $\sin(n\varphi)$ and $\cos(n\varphi)$ into polynomials in $\sin(\varphi)$ and $\cos(\varphi)$:

1. Write down De Moivre's Theorem for the appropriate value of n and for $r = 1$, i.e.

$$(\cos(\varphi) + i\sin(\varphi))^n = \cos(n\varphi) + i\sin(n\varphi). \tag{1.18}$$

2. Expand the left hand side directly, using the binomial expansion with coefficients from the Pascal triangle.
3. Replace all occurrences of i^2 by -1 and collect the real and imaginary parts.
4. By De Moivre, the real part on the left hand side is equal to $\cos(n\varphi)$, the imaginary part is equal to $\sin(n\varphi)$.
5. Often the expressions can be further simplified by using the identity $\sin^2\phi + \cos^2\phi = 1$.

1.6 Roots of Complex Numbers

- A complex number w is called an n-th root of another complex number z if

$$w^n = z \quad \Longleftrightarrow \quad w = z^{1/n} = \sqrt[n]{z}. \tag{1.19}$$

- Complex numbers have n distinct n-th roots which are given by

$$w_k = z^{1/n} = [r(\cos\varphi + i\sin\varphi)]^{1/n} = r^{1/n}\left[\cos\left(\frac{\varphi + 2\pi k}{n}\right) + i\sin\left(\frac{\varphi + 2\pi k}{n}\right)\right] \quad \text{for } n = 0, 1, ..., (n-1) \tag{1.20}$$

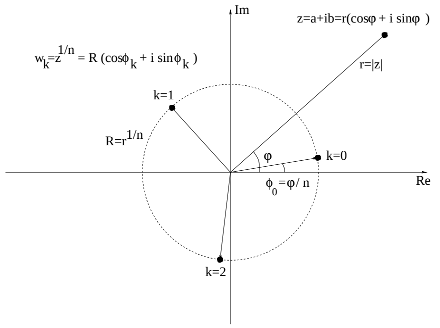

Figure 1.3: Graphical representation of the n-th root of a complex number $z = r(\cos(\varphi) + i\sin(\varphi))$, illustrated for $n = 3$.

- The geometrical interpretation of (1.20) in Fig. 1.3 shows that the n complex roots $w_k(k = 0, 1, ..., (n-1))$ are equally spaced on a circle of radius $R = r^{1/n}$ in the complex plane. The first root (corresponding to $k = 0$) has the argument $\phi_0 = \varphi/n$.

1.7 The complex exponential and Euler's relation

- The exponential of a complex number $z = a + ib$ is defined as

$$e^z = e^{a+ib} = e^a(\cos b + i\sin b). \tag{1.21}$$

Therefore, we have

$$|e^{a+ib}| = e^a \tag{1.22}$$

and

$$\arg(e^{a+ib}) = b \tag{1.23}$$

- For a purely imaginary argument we obtain *Euler's relation*

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{for real } \theta \tag{1.24}$$

which leads to an alternative representation of complex numbers in polar form:

$$z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}. \tag{1.25}$$

- Note that De Moivre's theorem is trivial in this notation:

$$z^n = [r(\cos\varphi + i\sin\varphi)]^n = (re^{i\varphi})^n = r^n e^{in\varphi} = r^n[\cos(n\varphi) + i\sin(n\varphi)]. \tag{1.26}$$

1.8 The complex logarithm

- A complex number w is called a logarithm of another complex number z if

$$z = e^w \quad \Longleftrightarrow \quad w = \log z. \tag{1.27}$$

- Complex numbers have infinitely many logarithms which are given by

$$\log z = \log[r(\cos\varphi + i\sin\varphi)] = \log r + i(\varphi + 2\pi k) \qquad \text{for } k = 0, 1, ... \tag{1.28}$$

- Again, we can restrict ourselves to the ***principal value*** in which the imaginary part lies between 0 and 2π.

1.9 The complex forms for $\sin, \cos, \sinh, \cosh$

- The following complex definitions for $\sin, \cos, \sinh, \cosh$ are motivated by Euler's relation:

$$\cos z = \cos(a + ib) = \frac{e^{iz} + e^{-iz}}{2} = \cos a \cosh b - i \sin a \sinh b \tag{1.29}$$

$$\sin z = \sin(a + ib) = \frac{e^{iz} - e^{-iz}}{2i} = \sin a \cosh b + i \cos a \sinh b \tag{1.30}$$

$$\cosh z = \cosh(a + ib) = \frac{e^{z} + e^{-z}}{2} = \cosh a \cos b + i \sinh a \sin b \tag{1.31}$$

$$\sinh z = \sinh(a + ib) = \frac{e^{z} - e^{-z}}{2} = \sinh a \cos b + i \cosh a \sin b \tag{1.32}$$

- $\sin z$ and $\cos z$ have period 2π, i.e. they have the same value whenever the real part of z increases by 2π.
- $\sinh z$ and $\cosh z$ have period $2\pi i$, i.e. they have the same value whenever the imaginary part of z increases by 2π.

1.10 Application for complex $\sin$ and $\cos$: Formulae for $\sin^n\theta$ and $\cos^n\theta$

The complex version of $\sin(z)$ and $\cos(z)$ can be used to derive expressions for $\sin^n\theta$ and $\cos^n\theta$ in terms of $\sin(n\theta)$ and $\cos(n\theta)$.

1. Raise the definition of the complex sin/cos to the n-th power:

$$\sin^n\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^n \qquad \text{or} \qquad \cos^n\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^n \tag{1.33}$$

2. Expand the right hand side using the binomial expansion with coefficients from the Pascal triangle.
3. Use Euler's relation to express exponentials in terms of sin and cos, i.e. $e^{im\theta} = \cos m\theta + i\sin m\theta$ for all m.
4. Test: Since the left hand side is real, the imaginary part of the right hand side has to vanish.
5. Now the right hand side only contains sin and cos of multiples of θ as required.