EXAMPLE SHEET VII

1) (i) $$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho F_i + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

where $F_1 = g \sin\alpha$, $F_2 = -g\cos\alpha$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$u_i = 0$ at $y = 0$

$$\underline{u} \to U \frac{y(2H-y)}{H^2} \underline{e}_x \quad \text{for } x \to \pm\infty$$

free surface: $f = h(x,t) - y = 0 \Rightarrow \frac{Df}{Dt} = 0$

$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} - v = 0$ at $y = h(x,t)$

& $h(x,t) \to H$ for $x \to \pm\infty$

$$-p n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j + \sigma\kappa n_i = 0$$

at $y = h(x,t)$

$u_i = 0$ at $x^2 + (y-b^2) = a^2$

(ii) Choose scales:

Length: a

veloc: U

pressure: $\frac{\mu U}{a}$ (viscous scale)

time: $\frac{a}{U}$ (no natural time scale in the problem)

body force: g

$u_i = U\tilde{u}_i$; $t = \tilde{t}\frac{a}{U}$; $p = \tilde{p}\frac{\mu U}{a}$

$x_i = a\tilde{x}_i$ $h = \tilde{h} a$; $F_i = g\tilde{F}_i$

$\kappa = \tilde{\kappa}\frac{1}{a}$ (free surface curvature)

$\underline{n}$ is already nondim. as it is a unit normal.

So:

N.St:

$$\frac{\rho U^2}{a}\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\mu U}{a^2}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \rho g \tilde{F}_i + \frac{\mu U}{a^2}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}$$

$$\underbrace{\frac{\rho U a}{\mu}}_{Re}\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \underbrace{\frac{\rho g a^2}{\mu U}}_{Gr}\tilde{F}_i + \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}$$

$$\boxed{Re\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + Gr\,\tilde{F}_i + \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}}$$

$$\frac{\partial u_i}{\partial x_i} = \frac{U}{a}\frac{\partial \tilde{u}_i}{\partial \tilde{x}_i} = 0$$

$$\boxed{\frac{\partial \tilde{u}_i}{\partial \tilde{x}_i} = 0}$$

$u_i = 0$ at $y = 0 = a\tilde{y}$

$$\boxed{\tilde{u}_i = 0 \quad \text{at } \tilde{y} = 0}$$

$$\underline{u} \to U y \frac{(2H-y)}{H^2} \underline{e_x} \quad \text{for } x \to \pm\infty$$

$$\cancel{U}\underline{\tilde{u}} \to \cancel{U} \frac{a\tilde{y}(2H-a\tilde{y})}{H^2} \underline{e_x} \quad \text{for } \tilde{x} \to \pm\infty$$

$$\boxed{\underline{\tilde{u}} \to \left(\frac{a}{H}\right)^2 \tilde{y}\left(2\frac{H}{a} - \tilde{y}\right) \quad \text{for } \tilde{x} \to \pm\infty}$$

Free surface: $\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} - v = 0$ at $y = h(x,t)$

$$\frac{\cancel{aU}}{\cancel{a}} \frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\cancel{Ua}}{\cancel{a}} \tilde{u} \frac{\partial \tilde{h}}{\partial \tilde{x}} + \cancel{U}\tilde{v} = 0 \quad \text{at } \cancel{a}\tilde{y} = \cancel{a}\tilde{h}(\tilde{x},\tilde{t})$$

$$\boxed{\frac{\partial \tilde{h}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{h}}{\partial \tilde{x}} - \tilde{v} = 0 \quad \text{at } \tilde{y} = \tilde{h}(\tilde{x},\tilde{t})}$$

$$-p n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j + \sigma \kappa n_i = 0$$

$$-\frac{\mu U}{a} \tilde{p} n_i + \frac{\mu U}{a}\left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i}\right) n_j + \frac{\sigma}{a} \tilde{\kappa} n_i = 0$$

$$-\tilde{p} n_i + \left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i}\right) n_j + \underbrace{\frac{\sigma}{\mu U}}_{\frac{1}{Ca}} \tilde{\kappa} n_i = 0$$

$$\boxed{-\tilde{p} n_i + \left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i}\right) n_j + \frac{1}{Ca} \tilde{\kappa} n_i = 0}$$

at $\tilde{y} = \tilde{h}(\tilde{x},\tilde{t})$

$u_i = 0$ at $x^2 + (y-b)^2 = a^2$

$\tilde{u}_i = 0$ at $a^2\tilde{x}^2 + (a\tilde{y} - b)^2 = a^2$

$$\boxed{\tilde{u}_i = 0 \quad \text{at} \quad \tilde{x}^2 + \left(\tilde{y} - \frac{b}{a}\right)^2 = 1}$$

$h(x,t) \to H$ for $x \to \pm\infty$

$$\boxed{\tilde{h}(\tilde{x},\tilde{t}) \to \frac{H}{a} \quad \text{for} \quad \tilde{x} \to \pm\infty}$$

3 The problem is indeed governed by three geometrical parameters $\frac{a}{b}$, $\frac{H}{a}$, α & the non-dim. parameters

$Re = \frac{aU\rho}{\mu}$ [ratio of inertial to viscous effects]

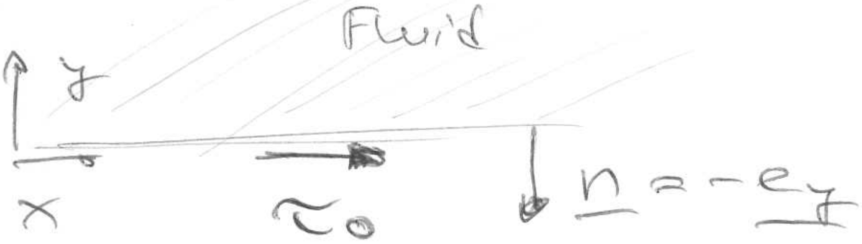
$Ca = \frac{\mu U}{\sigma}$ [ratio of viscous to surface tension effects]

$Gr = \frac{\rho g a^2}{\mu U}$ [ratio of body force to viscous effects].

2)

(i) $\boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}}$ for parallel flow

Stress B.C:



$n_1 = 0$
$n_2 = -1$

$t_i = \tau_{ij} n_j$; $i=2$: $\tau_0 = 0 + \Big(\overbrace{\frac{\partial u_2}{\partial x_1}}^{0} + \underbrace{\frac{\partial u_1}{\partial x_2}}_{\frac{\partial u}{\partial y}}\Big) \underbrace{n_2}_{-1}$

<u>BC:</u> $\boxed{\tau_0 = -\mu \frac{\partial u}{\partial y}}$

Also decay at ∞:

$\boxed{u \to 0 \quad \text{as } y \to +\infty}$

<u>IC:</u> $\boxed{u = 0 \quad \text{for } t = 0}$

(ii) The governing PDE is linear & homogeneous; the 1^{st} BC is inhomogeneous in τ_0. (all other cond. are homog.)

$\to$ The solution <u>must</u> be linear in τ_0.

$$u = \tau_0 \, F(y, t, \nu, \rho)$$

Now check the dimensions (e.g. using SI units)

$$[\mu] = \frac{kg}{m \, sec} \quad \text{or} \quad [\rho] = \frac{kg}{m^3}$$

Specification of one is suff. since $\nu = \frac{\mu}{\rho}$

$$[u] = \frac{m}{sec}$$

$$[\tau_0] = \frac{kg}{m \, sec^2}$$

$$[\nu] = \frac{m^2}{sec}$$

$$[y] = m$$

with these parameters, the easiest way to get rid of the kg in $[\tau_0]$ is to divide by ρ

$$u = \frac{\tau_0}{\rho} \, G(y, t, \nu, \rho)$$

G can still depend on ρ as u does not have to be linear in ρ!

$$\left[\frac{\tau_0}{\rho}\right] = \frac{\cancel{kg} \; \cancel{m^3}^{2}}{\cancel{m} \, sec^2 \, \cancel{kg}} = \frac{m^2}{sec^2}$$

So, to make this dimensionally coherent we need to divide by a comb. of parameters which have the dim. of a velocity. Can choose either $\frac{y}{t}$ (but this would spoil the similarity form – we don't want y outside the function!) or $\sqrt{\frac{\nu}{t}}$.

So:

$$u = \underbrace{\frac{\tau_0}{\rho}\sqrt{\frac{t}{\nu}}}_{\text{units of veloc.}} \; \underbrace{f(y, t, \nu, \rho)}_{\text{DIM. LESS!}}$$

Since f is dim.less, its arguments have to appear in a dimensionless combination. The kg in ρ cannot be cancelled by any other quantity [τ_0 has to remain "outside" f to conserve the linearity].

Hence

$$f(y,t,\nu,\rho) = f(y,t,\nu)$$

$$= f\Big(\underbrace{\frac{y}{\sqrt{\nu t}}}_{\eta}\Big)$$

as in the impulsively started plate problem. (Powers of η would be alternative similarity variables but one usually aims to have them linear in the spatial coordinate).

So:

$$u = \frac{\tau_0}{\rho}\sqrt{\frac{t}{\nu}}\, f\Big(\frac{y}{\sqrt{\nu t}}\Big)$$

Now work out derivatives:

$$\frac{\partial u}{\partial t} = \frac{\tau_0}{\rho\sqrt{\nu}}\Big(\frac{1}{2}t^{-1/2} f + t^{1/2} f' \frac{y}{\sqrt{\nu}}\Big(-\frac{1}{2}\Big)t^{-3/2}\Big)$$

$$\frac{\partial u}{\partial t} = \frac{\tau_0}{\rho\sqrt{\nu}}\Big(\frac{1}{2}t^{-1/2} f - \frac{1}{2}\frac{y}{\sqrt{\nu}}\frac{1}{t} f'\Big)$$

$$\frac{\partial u}{\partial y} = \frac{\tau_0}{\rho}\sqrt{\frac{t}{\nu}}\,\frac{1}{\sqrt{\nu t}}\,f'$$

$$= \frac{\tau_0}{\underbrace{\rho\nu}_{\mu}}\,f'$$

$$\frac{\partial u}{\partial y} = \frac{\tau_0}{\mu}\,f'$$

and:

$$\frac{\partial^2 u}{\partial y^2} = \underbrace{\frac{\tau_0}{\rho}\sqrt{\frac{t}{\nu}}}_{\mu}\,\frac{1}{\nu t}\,f'' = \frac{\tau_0}{\mu\sqrt{\nu t}}\,f''$$

Into PDE:

$$\frac{\partial u}{\partial t} = \nu\,\frac{\partial^2 u}{\partial y^2}$$

$$\frac{1}{2}\,\frac{\cancel{\tau_0}}{\rho\sqrt{\nu}}\left(t^{-1/2} f - \frac{y}{\sqrt{\nu}\,t}\,f'\right) = \frac{\underbrace{\nu}_{\frac{1}{\rho}}\,\cancel{\tau_0}}{\mu\sqrt{\nu t}}\,f''$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{\nu t}}\,f - \frac{y}{\nu t}\,f'\right) = \frac{1}{\sqrt{\nu t}}\,f''$$

$$2f'' + \frac{y}{\sqrt{\nu t}}\,f' - f = 0$$

$$\boxed{2f'' + \eta f' - f = 0}$$

So PDE can be transformed into ODE in η.

what about BC/IC ?

$\tau_0 = -\mu \frac{\partial u}{\partial y}$ insert $\frac{\partial u}{\partial y} = \frac{\tau_0}{\mu} f'$

$$\boxed{f' = -1 \quad \text{at } y = 0 \; ; \; \eta = 0}$$

$u \to 0$ for $y \to \infty$

$$\boxed{f \to 0 \quad \text{for } \eta \to \infty}$$

IC:

$u = 0$ at $t = 0$. is consistent with the 2 transformed B.C. as $u \sim \sqrt{t}\, f(\eta)$ & $\eta \to \infty$ for $t \to 0$ & $f \to 0$ as $\eta \to \infty$.

So 2 BC. for the second order ODE in η fulfill the 3 cond. in the PDE ☺ Similarity works!

(iii) By inspection we see that

$f_1 = \eta$ does indeed solve

the ODE (but does not fulfill the BC).

Recall: (2nd year?)

$$y''(x) + p(x)\, y'(x) + q(x)\, y(x) = f(x)$$

If one soln y_1 is known, a second one can be constructed via

$$y_2(x) = A\, y_1(x) \int^{x} \frac{1}{y_1(t)^2} \exp\left(-\int^{t} p(s)\, ds\right) dt$$

Here:

$$f'' + \frac{1}{2}\eta f' - \frac{1}{2} f = 0$$

$$p(\eta) = \frac{1}{2}\eta$$

$$\int_0^t p(s)\, ds = \frac{1}{4} t^2$$

arbitrary; 0 is most convenient

$$f_2(\eta) = \hat{A}\,\eta \int_{\infty}^{\eta} \frac{\exp(-\frac{1}{4}t^2)}{t^2}\,dt$$

"Lower" limit is again arbitrary, but ~~The~~ at least the integral should exist in that limit: $\eta = +\infty$ is a convenient choice.

So the general soln is given by

$$\boxed{f(\eta) = A\eta \int_{\eta}^{\infty} \frac{\exp(-\frac{1}{4}t^2)}{t^2}\,dt + B\eta}$$

(iv) A & B from the BC.

$f(\eta) \to 0$ for $\eta \to \infty$

requires $B = 0$

So:

$$f(\eta) = \hat{A}\,\eta \int_{\infty}^{\eta} \frac{\exp(-\frac{1}{4}t^2)}{t^2}\,dt$$

integrate the integral by parts

$$\int_{\infty}^{\eta} \underbrace{t^{-2}}_{f'} \underbrace{\exp(-\tfrac{1}{4}t^2)}_{g}\, dt =$$

$$\left[f = -\frac{1}{t} \quad g' = -\frac{1}{2} t \exp(-\tfrac{1}{4}t^2) \right]$$

$$= \left[-\frac{1}{t} \exp(-\tfrac{1}{4}t^2) \right]_{\infty}^{\eta} - \int_{\infty}^{\eta} \left(-\frac{1}{t}\right)\left(-\frac{1}{2} t \exp(-\tfrac{1}{4}t^2)\right) dt$$

$$= -\frac{\exp(-\frac{1}{4}\eta^2)}{\eta} - \int_{\infty}^{\eta} \frac{1}{2} \exp(-\tfrac{1}{4}t^2)\, dt$$

So:

$$f(\eta) = \hat{A}\left(-\exp(-\tfrac{1}{4}\eta^2) - \eta \int_{\infty}^{\eta} \frac{1}{2} \exp(-\tfrac{1}{4}t^2)\, dt \right)$$

Now:

$$\left.\frac{\partial f}{\partial \eta}\right|_{\eta=0} = -1$$

$$\frac{\partial f}{\partial \eta} = \hat{A}\left(\frac{1}{2}\eta \exp(-\tfrac{1}{4}\eta^2) - \frac{\eta}{2} \exp(-\tfrac{1}{4}\eta^2) + \frac{1}{2} \int_{\eta}^{\infty} \exp(-\tfrac{1}{4}t^2)\, dt \right)$$

$$\left.\frac{\partial f}{\partial \eta}\right|_{\eta=0} = \frac{\hat{A}}{2} \underbrace{\int_0^\infty \exp(-\tfrac{1}{4}t^2)\,dt}_{s=\frac{t}{2} \quad dt = 2s}$$

$$\int_0^\infty \exp(-\tfrac{1}{4}t^2)\,dt = 2\underbrace{\int_0^\infty \exp(-s^2)\,ds}_{\frac{\sqrt{\pi}}{2}} = \sqrt{\pi}$$

$$-1 = \frac{\sqrt{\pi}}{2}\hat{A} \qquad \hat{A} = -\frac{2}{\sqrt{\pi}}$$

$$f(\eta) = \frac{2}{\sqrt{\pi}}\left\{\exp(-\tfrac{1}{4}\eta^2) - \frac{\eta}{2}\int_\eta^\infty \exp(-\tfrac{1}{4}t^2)\,dt\right\}$$

Can also be written in terms of the error fct...

Addendum to Q2 on Example Sheet VII

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = -\frac{\tau_0}{\mu} \qquad (*)$$

$u \to 0$ as $y \to \infty$

$u = 0$ for $t = 0$

Let's interpret $(*)$ as

$$u(y, t; \nu, \mu, \tau_0) = \frac{\tau_0}{\mu} F(y, t; \nu)$$

Now F must not depend on τ_0 or μ

Check dimensions:

$$\left[\frac{\tau_0}{\mu}\right] = \frac{1}{sec}$$

so to make

$$u = \frac{\tau_0}{\mu} f(y, t; \nu)$$

dimensionally consistent have to combine the arguments of f to something with dimension m.

$$[y] = m$$

$$[t] = sec$$

$$[\nu] = \frac{m^2}{sec}$$

<u>Choice 1:</u>

$$u = \underbrace{\frac{\tau_0}{\mu} y}_{\text{spoils the similarity form}} f(y, t; \nu)$$

<u>Choice 2:</u>

$$u = \frac{\tau_0}{\mu} \sqrt{\nu t}\, f(y, t; \nu)$$

$$u = \frac{\tau_0 \sqrt{t} \sqrt{\nu}}{\nu \rho} f(y, t; \nu)$$

$$u = \frac{\tau_0}{\rho} \sqrt{\frac{t}{\nu}}\, f(y, t; \nu)$$

as before....