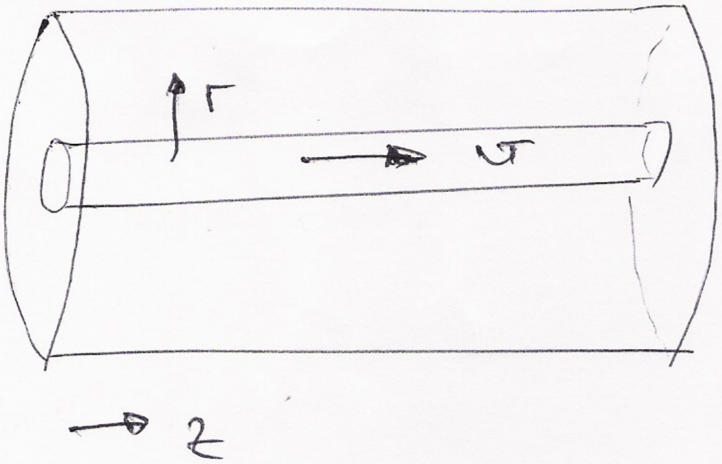
EXAMPLE SHEET VI

1)



steady, φ & z - indep. B.C.

$\Rightarrow$ expect (try!) solution:

$$\underline{u} = (0, 0, w) = w(r)\, \underline{e_z}$$

$p = p(z)$ (from r & φ momentum eqns)

$\Rightarrow$ only z-momentum eqn is nonzero & simplifies to:

$$\underbrace{\frac{\partial p}{\partial z}}_{G} = \mu \nabla^2 w$$

$$\frac{1}{\mu} G = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r}\right) = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}$$

$$\frac{G}{\mu} r = \frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r}\right)$$

$$\frac{1}{2}\frac{G}{\mu} r^2 + \tilde{A} = r \frac{\partial w}{\partial r}$$

$$\frac{1}{2}\frac{G}{\mu} r + \frac{\tilde{A}}{r} = \frac{\partial w}{\partial r}$$

$$w = \frac{1}{4}\frac{G}{\mu} r^2 + \tilde{A} \ln r + \tilde{B}$$

$$w = \frac{1}{4}\frac{G}{\mu} r^2 + A \ln\left(\frac{r}{b}\right) + B$$

B.C:

$w(r=b) = 0$

$$B = -\frac{1}{4}\frac{G}{\mu} b^2$$

$w(r=a) = U$

$$U = \frac{1}{4}\frac{G}{\mu}(a^2 - b^2) + A \ln\left(\frac{a}{b}\right)$$

$$A = \frac{+U}{\ln(a/b)} + \frac{\frac{1}{4}\frac{G}{\mu}(b^2 - a^2)}{\ln\left(\frac{a}{b}\right)}$$

$$w(r) = \frac{1}{4}\frac{G}{\mu}\left[(r^2-b^2) + \frac{(b^2-a^2)\ln\left(\frac{r}{b}\right)}{\ln\left(a/b\right)}\right] + U\,\frac{\ln(r/b)}{\ln(a/b)}$$

Drag:

$$D = \underbrace{\tau_w}_{\text{wall shear}} \cdot \underbrace{\underbrace{2\pi a}_{\text{circumfer.}} \cdot \underbrace{L}_{\text{length.}}}_{\text{wetted area.}}$$

$$\tau_w = t_z$$

$$t_i = \tau_{ij} n_j \qquad \underline{n} = -\underline{e_r}$$

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} \qquad i = 1,2,3 \leftrightarrow i = r, \varphi, z$$

$$n_r = -1 \quad n_\varphi = 0 \quad n_z = 0$$

$$t_z = \underbrace{\tau_{zj} n_j}_{\text{only } j=r \text{ survives}}$$

$$t_z = -2\mu e_{zr} = -\mu\left(\cancel{\frac{\partial u}{\partial z}} + \frac{\partial w}{\partial r}\right) = -\mu\frac{\partial w}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{1}{4}\frac{G}{\mu}\left[2r + \frac{b^2-a^2}{r\ln(a/b)}\right] + \frac{U}{r\ln(a/b)}$$

$$\frac{D}{L} = 2\pi a\, \tau_w\Big|_{r=a}$$

drag per unit length

$$= -\left(\frac{\pi}{2} Ga\left[2a + \frac{b^2-a^2}{a\ln(a/b)}\right] + \frac{2\pi\mu U}{\ln(a/b)}\right)$$

(ii) $\frac{D}{L} = 0$ for

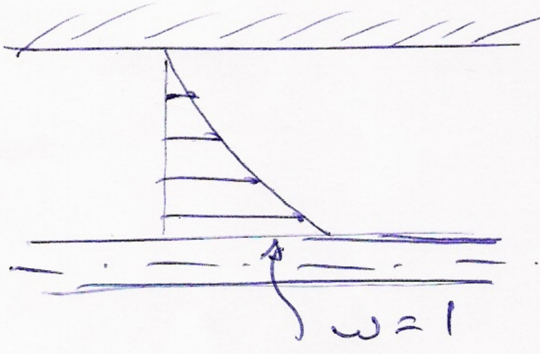
$$G = G_0 = -\frac{\cancel{2\pi}\mu U}{\ln(a/b)\,\frac{\pi}{2}a\left[2a + \frac{b^2-a^2}{a\ln(a/b)}\right]}$$

$$G_0 = -\frac{4\mu U}{a}\frac{1}{2a\ln(a/b) + \frac{b^2-a^2}{a}}$$

$$G_0 = -\frac{4\mu U}{a^2}\frac{1}{2\ln(a/b) + \left(\frac{b}{a}\right)^2 - 1}$$

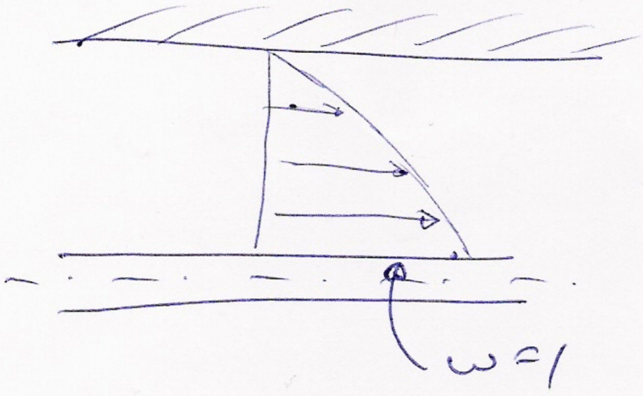
velocity profiles:

$G = 0$



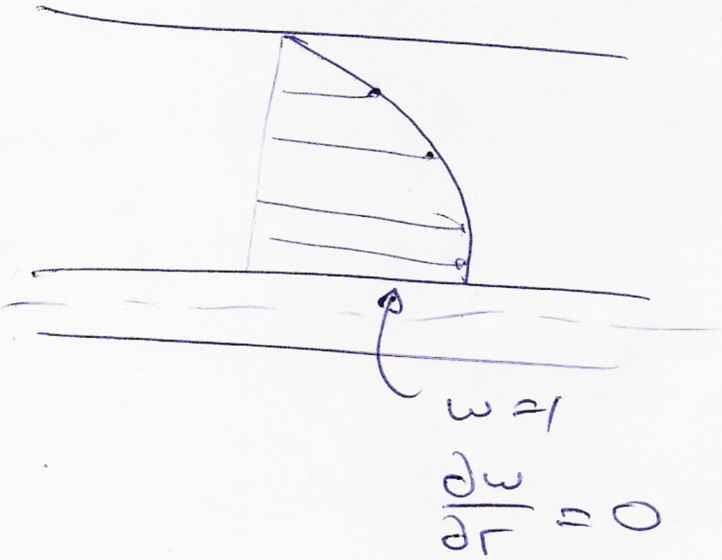
Flow only driven by boundary motion

$G < 0$



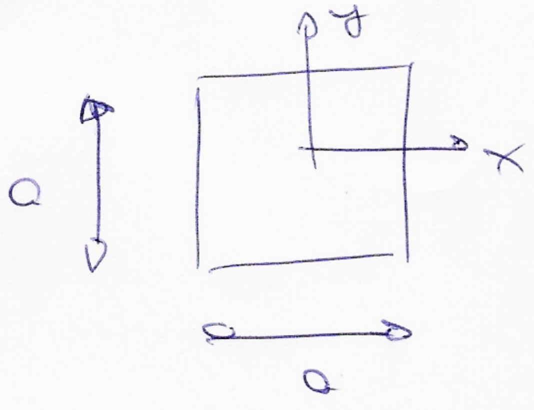
Combination of boundary & pressure driven flow [flow against press. gradient!]

$G = G_0 < 0$



Critical case. No drag.

2) parallel steady flow



$$\boxed{\begin{gathered}\frac{G}{\mu} = \nabla^2 w \\ w\big|_{x=\pm a/2} = w\big|_{y=\pm a/2} = 0\end{gathered}}$$

$$w = w_p(y) + w_H(x,y)$$

Choose w_p s.th. it fulfills the BC.

$$w_p(y) = \frac{1}{2}\frac{G}{\mu}\left(y^2 - \left(\frac{a}{2}\right)^2\right)$$

$$\nabla^2 w_p = \frac{G}{\mu} \qquad \& \; w_p(y = \pm a/2) = 0$$

Then homogeneous problem

$$\nabla^2 w_H = 0$$

Sep. of variables:

$$w_H(x,y) = X(x)\,Y(y)$$

$$\nabla^2 w_H = X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{const.} = +\lambda^2$$

$\Rightarrow$ 2 ODEs:

$$Y'' + \lambda^2 Y = 0$$

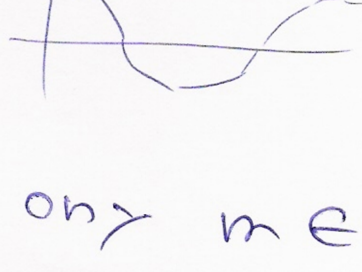
$$X'' - \lambda^2 X = 0$$

$$Y(y) = A\sin(\lambda y) + B\cos(\lambda y)$$

BC: $w_H\big|_{y=\pm a/2} = 0$

& symmetric soln. only $\rightarrow A = 0$

$$\cos\left(\lambda \frac{a}{2}\right) = 0$$



$\frac{a}{2}\lambda = (2m-1)\frac{\pi}{2}$ for any $m \in \mathbb{N}$

$$\lambda_m = (2m-1)\frac{\pi}{a}$$

Also:

$$X(x) = C e^{\lambda x} + D e^{-\lambda x}$$

Symmetry demands $X(x) = X(-x)$

$$C = D$$

$$X(x) = C\left(e^{\lambda x} + e^{-\lambda x}\right) = 2C \cosh \lambda x$$

So the complete soln. is given by

$$w_H \approx \cosh(\lambda_m x) \cos(\lambda_m y)$$

& since the eqn. is linear this holds for arbitrary linear combinations of these fcts. for all values of m.

$$w_H(x,y) = \sum_{m=1}^{\infty} A_m \cosh(\lambda_m x) \cos(\lambda_m y)$$

for $\lambda_m = (2m-1) \frac{\pi}{a}$

$$\Big\} \; w = w_p + w_H$$

$$w = \frac{1}{2}\frac{G}{\mu}\left(y^2 - \left(\frac{a}{2}\right)^2\right) + \sum_{m=1}^{\infty} A_m \cosh(\lambda_m x)\cos(\lambda_m y)$$

The B.C. @ $y = \pm a/2$ are already fulfilled thanks to our clever choice of the particular soln. & the choice of λ_m.

The soln. is symmetric in x so we only need to fulfill

$$w(y, x = \tfrac{a}{2}) = 0$$

This provides an eqn. for the A_m.

$$\frac{1}{2}\frac{G}{\mu}\left(\left(\frac{a}{2}\right)^2 - y^2\right) = \sum_{m=1}^{\infty} A_m \cosh\left(\lambda_m \frac{a}{2}\right)\cos(\lambda_m y)$$

Fourier expand LHS:

$$\frac{1}{2}\frac{G}{\mu}\left(\left(\frac{a}{2}\right)^2 - y^2\right) = \sum_1 B_m \cos\left((2m-1)\frac{\pi}{a}y\right)$$

$$\int_{-a/2}^{a/2} \cos\left((2m-1)\frac{\pi}{a}y\right)\cos\left((2n-1)\frac{\pi}{a}y\right)dy$$

$$= \begin{cases} \frac{a}{2} & \text{for } m = n \\ 0 & \text{for } m \neq n. \end{cases}$$

$$\int_{-a/2}^{a/2} \frac{1}{2}\frac{G}{\mu}\left(\left(\frac{a}{2}\right)^2 - y^2\right)\cos\left((2m-1)\frac{\pi}{a}y\right)dy$$

(opposite sign to result given)

$$= -\frac{2a^3 G(-1)^m}{(2m-1)^3\pi^3\mu} = \frac{a}{2}B_m$$

$$B_m = -\frac{4a^2 G(-1)^m}{(2m-1)^3\pi^3\mu}$$

$$\sum_1 B_m \cos(\lambda_m y) = \sum_1 A_m \cosh\left(\lambda_m \frac{a}{2}\right) \cos(\lambda_m y)$$

compare coeffns:

$$A_m = \frac{B_m}{\cosh\left(\lambda_m\left(\frac{a}{2}\right)\right)}$$

$$w = \frac{1}{2}\frac{G}{\mu}\left(y^2 - \left(\frac{a}{2}\right)^2\right) - \sum_{m=1}^{\infty} \frac{4a^2 G(-1)^m \cosh\left((2m-1)\frac{\pi}{a}x\right)}{(2m-1)^3\pi^3\mu \cosh\left((2m-1)\frac{\pi}{2}\right)}$$

$$\times \cos\left((2m-1)\frac{\pi}{a}y\right).$$