BC

$\underline{u} = \underline{u}_0$ given

$\underline{u} = \underline{0}$

↳ No slip/no penetration

Free surface:

- Kinematic BC
- traction (dynamic) BC

(a) Kinematic BC

The posn. of the free surface can always be described implicitly as

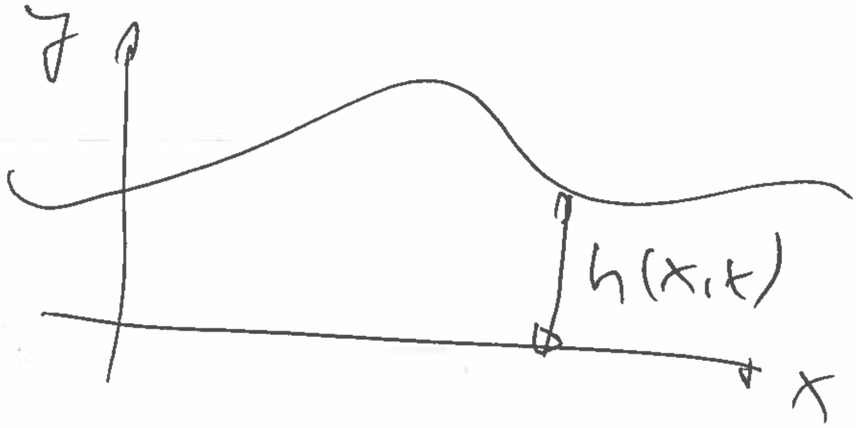
$$f(x, y, z, t) = 0$$

At least locally this can be inverted to give

$$z = h(x, y, t)$$

or in 2D

$$y = h(x, t)$$



Physics: fluid particles on the free surface stay on that free surface.

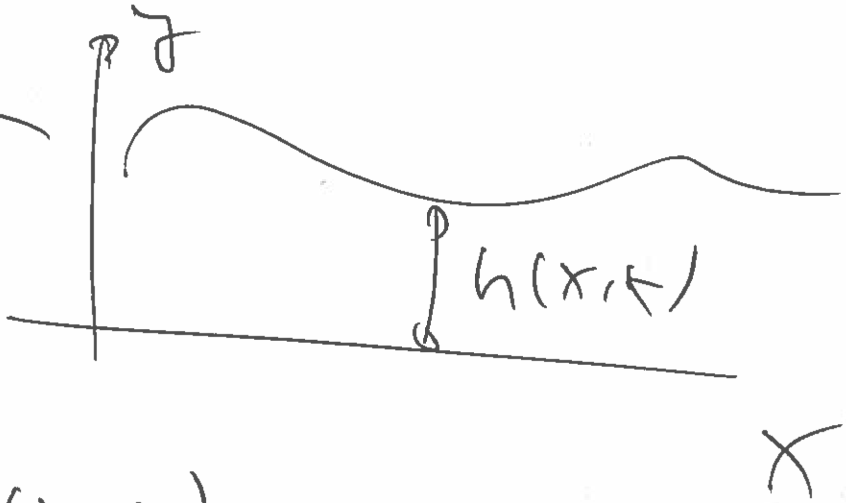
$\Rightarrow f(x_i, t) \equiv 0$ for all fluid particles on the surface

$$\Rightarrow \boxed{\frac{Df}{Dt} = 0}$$

$$\boxed{\frac{\partial F}{\partial t} + u_j \frac{\partial F}{\partial x_j} = 0}$$

Evolution eqn. for F
as a fct. of $\underline{u}$.

Example:



$$y\big|_{\partial D} = h(x,t)$$

So choose e.g.:

$$F(x,y,t) = h(x,t) - y \equiv 0$$

for $y = h(x,t)$

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0$$

$$\frac{\partial f}{\partial t} = \frac{\partial h}{\partial t}$$

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x}$$

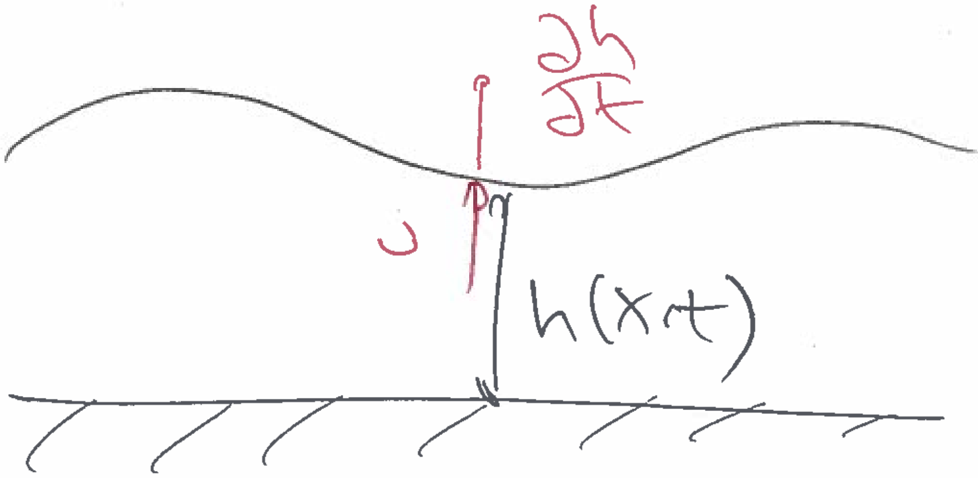
$$\frac{\partial f}{\partial y} = -1$$

$$\frac{Df}{Dt} = \boxed{\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} - v = 0}$$

Special case 1:

$u = 0$ (only vertical veloc.)

$$\frac{\partial h}{\partial t} = v$$



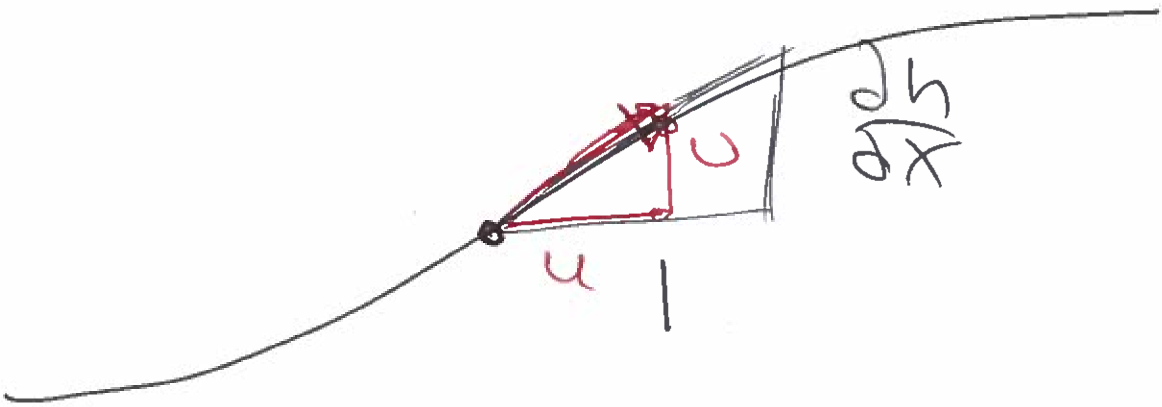
Special case 2:

Fixed free surface posn.

$$\frac{\partial h}{\partial t} = 0$$

$$u \frac{\partial h}{\partial x} = v$$

$$\frac{\partial h}{\partial x} = \frac{v}{u}$$

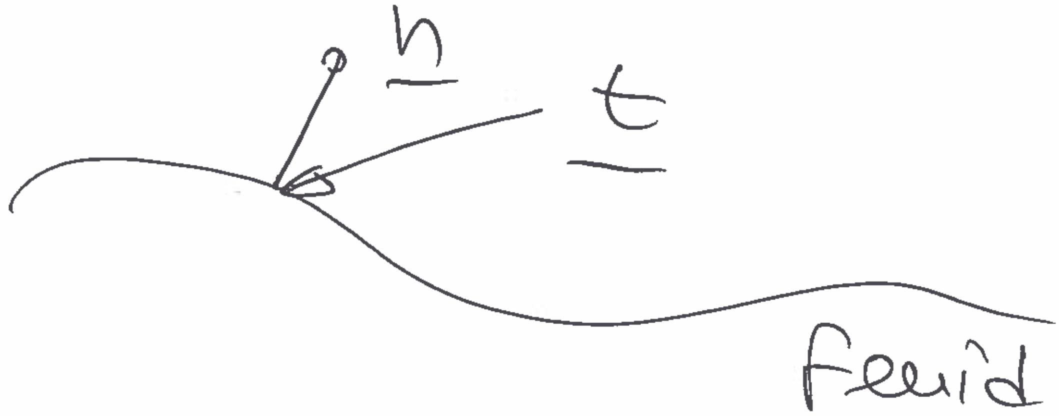


Veloc. is tangential to the surface!

(b) Traction / dynamic BC

Stress across a free surface must be continuous

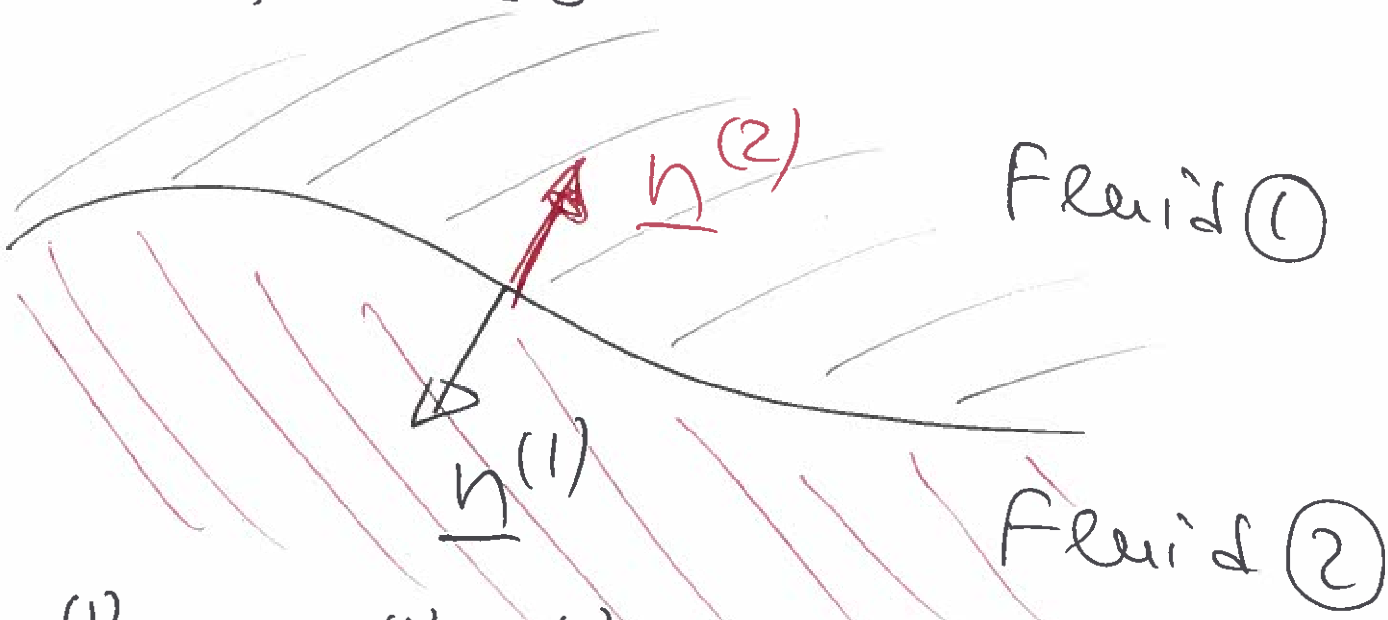
$\Longrightarrow$ stress = traction



$$\tau_{ij} n_j = t_i$$

stress tensor — outer unit normal — applied traction.

For a surface separating two fluids



$t_i^{(1)} = \tau_{ij}^{(1)} n_j^{(1)}$ traction <u>on</u> fluid 1.

$t_i^{(2)} = \tau_{ij}^{(2)} n_j^{(2)}$ — " — fluid 2.

Now: Action = reaction

$$\underline{t}^{(1)} = -\underline{t}^{(2)}$$

Also: $\underline{n}^{(1)} = -\underline{n}^{(2)}$

$$\boxed{\tau_{ij}^{(1)} n_j = \tau_{ij}^{(2)} n_j}$$

For any one outer unit normal on the surface.

<u>Example:</u> hydrostatics (or inviscid fluids)

$$\tau_{ij}^{(1,2)} = -p^{(1,2)} \delta_{ij}$$

$$-p^{(1)} \delta_{ij} n_j = -p^{(2)} \delta_{ij} n_j$$

$$p^{(1)} \underline{n} = p^{(2)} \underline{n}$$

$$\left(p^{(1)} - p^{(2)}\right) \underline{n} = \underline{0}$$

$$p^{(1)} = p^{(2)}$$

In practice: Also have surface tension. See notes or www.