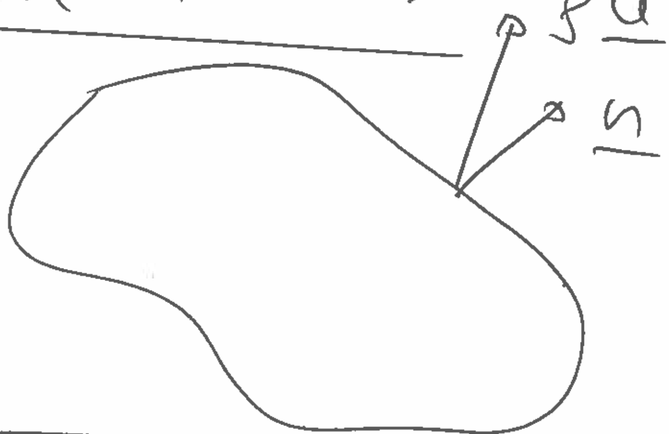
Continuity eqn

(mass conservation)

integral form:



$$\boxed{\int \frac{\partial \rho}{\partial t}\, dV = -\oint \rho \underline{u} \cdot \underline{n}\, dA}$$

$\rho = \rho(\underline{x}, t) =$ density $\left(\mathrm{kg}/\mathrm{m}^3\right)$

differential form:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

$$\underline{\frac{\partial \rho}{\partial t}} + \rho \frac{\partial u_j}{\partial x_j} + \underline{u_j \frac{\partial \rho}{\partial x_j}} = 0$$

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_j}{\partial x_j} = 0$$

incompressible fluids (are fluids whose density is constant): $\frac{D\rho}{Dt} = 0$

For those fluids:

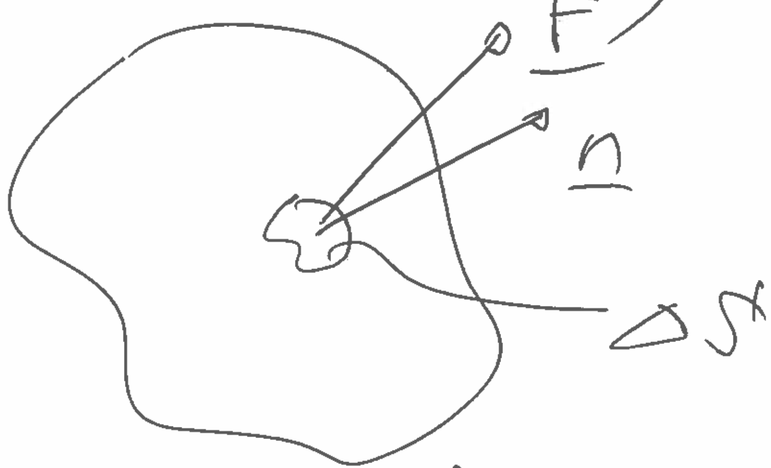
$$\boxed{\begin{gathered} \frac{\partial u_j}{\partial x_j} = 0 \\ \operatorname{div} \underline{u} = \nabla \cdot \underline{u} = 0 \end{gathered}}$$

This is a purely kinematic constraint!

§3 Stress, Cauchy's eqn. & the Navier Stokes eqns.

① The concept of stress/traction

Consider a finite blob of fluid loaded by some distributed force (pressure, sheer stress, etc)



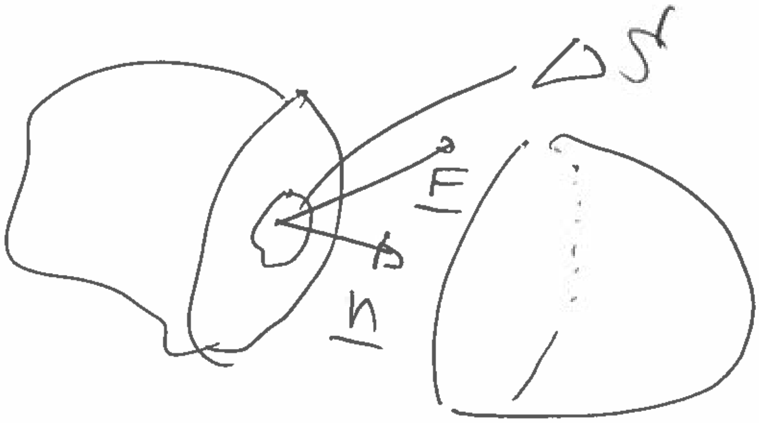
Every patch ΔS on the surface with outer unit normal $\underline{n}$ is subject to a resultant force $\underline{F}$

Def: Traction

$$\underline{t} = \lim_{\Delta S \to 0} \frac{\underline{F}}{\Delta S} \quad \text{vector}$$

$\underline{t}$ is traction exerted onto fluid.

Similarly: Cut the blob along a plane with normal $\underline{n}$:



Here $\underline{F}$ is the force exerted onto ΔS by "the other half" of the blob.

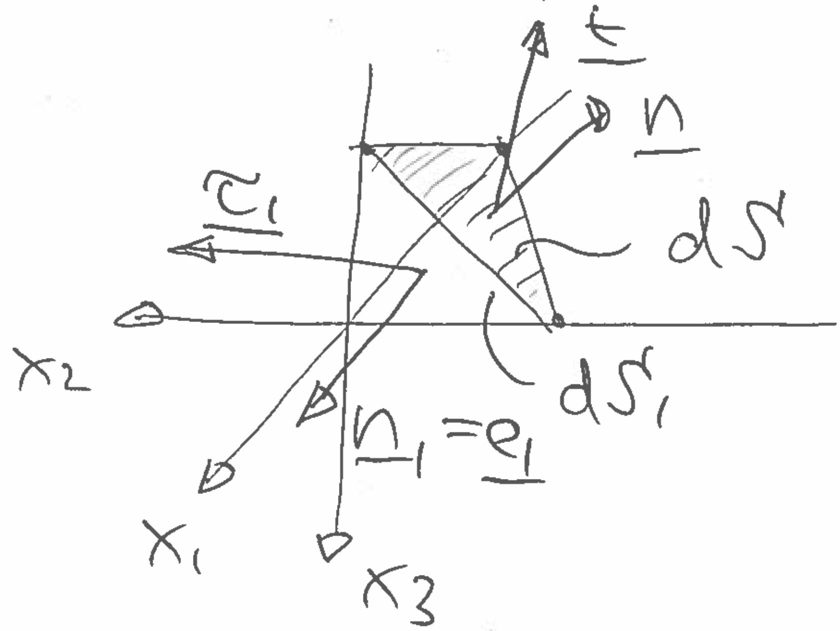
Def: Stress

$$\underline{t} = \lim_{\Delta S \to 0} \frac{\underline{F}}{\Delta S}$$

<u>Note:</u> The stress depends on
- position in the fluid
- the direction of the (imaginary) cut, $\underline{n}$.

② The stress tensor

To examine the dependence on $\underline{n}$ consider an infinitesimal tetrahedron:



Face i is characterised by $x_i = \text{const}$ & outer unit normal $\underline{n}_i = \underline{e}_i$

Represent faces (orientation & area) by area vectors: parallel to outer unit normal & length = area.

<u>Then:</u> Sum of all area vectors $= \underline{0}$.

Here:

$$\underline{n_i}\, dS_i + \underline{n}\, dS = \underline{0}$$

(EXERCISE)

multiply by $\underline{n_j} = \underline{e_j}$:

$$\underbrace{\underbrace{\underline{n_i} \cdot \underline{n_j}}_{\underline{e_i} \cdot \underline{e_j}}}_{\delta_{ij}} dS_i + \underline{n} \cdot \underline{e_j}\, dS = 0$$

$$dS_j + n_j\, dS = 0$$

$$\boxed{dS_j = -n_j\, dS}$$

<u>Now:</u> Balance of forces acting onto the tetrahedron:

$$\underline{t}\, dS = -\underline{\tau}_j\, dS_j$$

$$\underline{t}\, \cancel{dS} = \underline{\tau}_j\, n_j\, \cancel{dS}$$

in index notation:

$$t_i = \tau_{ij}\, n_j$$

τ_{ij} = i-th component of $\underline{\tau}_j$

= the stress tensor!