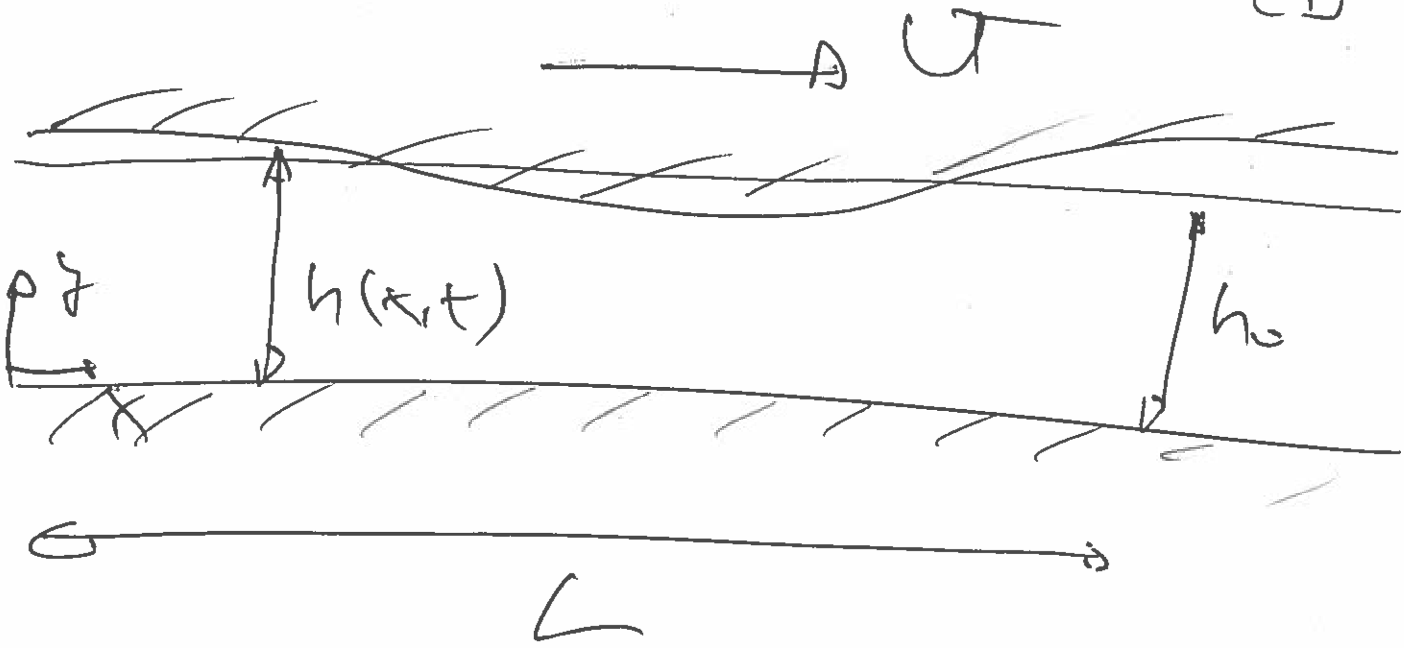
Further example for scaling

§9 Lubrication theory

2D



Gap narrow & gently varying

$h_0 << L$; $\frac{h_0}{L} << 1$

Scale:

$x = L\tilde{x}$ $\qquad$ $t = \frac{L}{U}\tilde{t}$

$y = h_0\tilde{y}$

$u = U\tilde{u}$ $\qquad$ $p = P\tilde{p}$

$v = V\tilde{v}$ $\qquad$ P & V unknown.

Continuity:

$$\frac{U}{L}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{V}{h_0}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$U\frac{h_0}{L}\frac{\partial \tilde{u}}{\partial \tilde{x}} + V\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

These terms only balance if

$$V = U\frac{h_0}{L}$$

so $V \ll U$ (as expected)

N. St:

$$\rho\left(\frac{U^2}{L}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{U^2}{L}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{U^2\cancel{h_0}}{L\cancel{h_0}}\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right) =$$

$$= -\frac{P}{L}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \mu\left(\frac{U}{L^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{U}{h_0^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right)$$

$$\frac{\rho U^2}{L}\frac{\partial \tilde{u}}{\partial \tilde{t}} = -\frac{P}{L}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\mu U}{h_0^2}\left(\cancel{\left(\frac{h_0}{L}\right)^2\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right)$$

$Re\left(\frac{h_0}{L}\right) \ll 1$

$$\cancel{\frac{\rho U h_0}{\mu}\left(\frac{h_0}{L}\right)\frac{\partial \tilde{u}}{\partial \tilde{t}}} = -\frac{P}{\left(\frac{\mu U}{h_0}\right)}\left(\frac{h_0}{L}\right)\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

Re based on h_0.

Note that in this geometry it is the product of Re & $\frac{h_0}{L}$ that determines the size of the non-lin. terms. They can be neglected if

$$Re\left(\frac{h_0}{L}\right) \ll 1.$$

Balancing the remaining terms determines P.

$$P = \frac{\mu U}{h_0} \frac{1}{\left(\frac{h_0}{L}\right)}$$

$$\boxed{0 = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}}$$

Similarly y-comp. N.St:

$$\boxed{\frac{\partial \tilde{p}}{\partial \tilde{y}} = 0}$$

(EXERCISE)

or in dimensional form

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$0 = -\frac{\partial p}{\partial y} \quad (2)$$

Parallel flow eqns!

The gentle slope of the wall means that locally the flow behaves like parallel flow through an infinitely long channel of the same width.

<u>BC:</u> $u(y=0)=0$

$u(y=h(x,t))=U$

Since (2): $p(x,y)=p(x)$

into (1):

$$\frac{\partial p}{\partial x}=\mu\frac{\partial^2 u}{\partial y^2}$$

$$u=\frac{1}{2\mu}\frac{\partial p}{\partial x}y^2+Ay+B \quad (+\text{ BC}):$$

$h(x,t)$

$$u(x,y,t)=\underbrace{\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2-hy\right)}_{\text{pressure-driven}}+\underbrace{U\frac{y}{h}}_{\text{shear flow}}$$

Also: what is $\frac{\partial p}{\partial x}$?