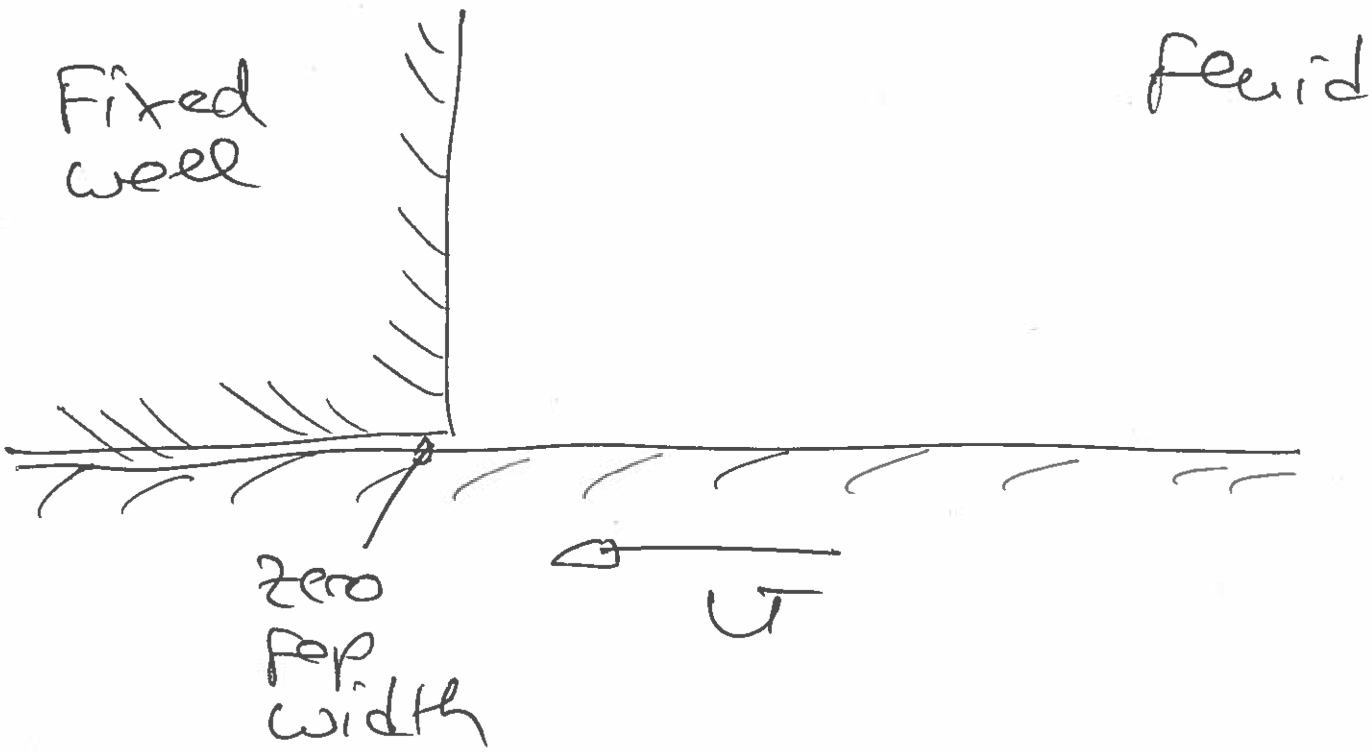
Stokes flow 2D

(using stream fct.)

Example: Scraping flow



Assume: Slow, steady viscous flow

⟹ Stokes eqns valid because $Re \ll 1$

$$D^4 \psi = 0$$

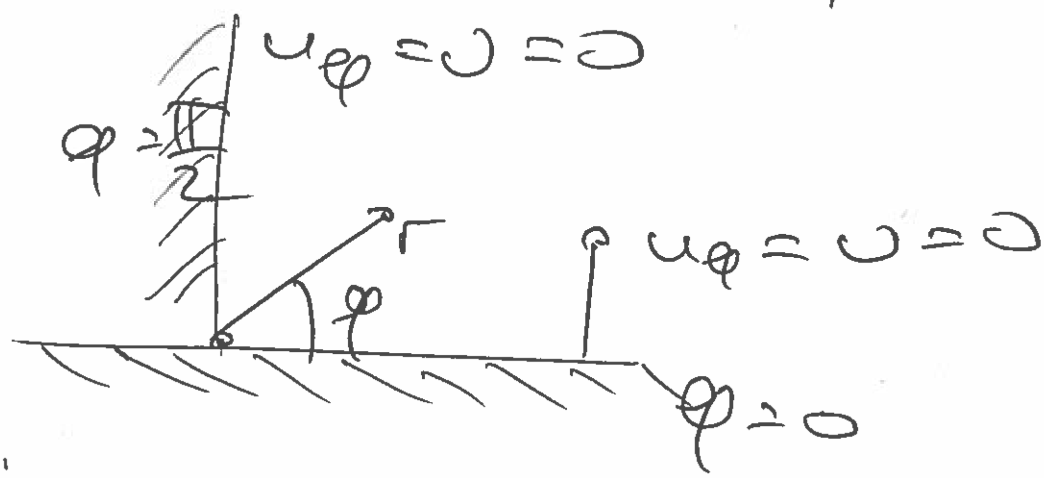
Coord system: polars!

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

$$u = u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi}$$

$$v = u_\varphi = -\frac{\partial \psi}{\partial r}$$

<u>BC:</u> Impermeability:



So:

$v = -\frac{\partial \psi}{\partial r} = 0$ at $\varphi = 0$ $\forall r$

$\psi = \text{const} = C_1$ at $\varphi = 0$

$v = -\frac{\partial \psi}{\partial r} = 0$ at $\varphi = \frac{\pi}{2}$ $\forall r$

$\psi = \text{const} = C_2$ at $\varphi = \pi/2$

Relation between C_1 & C_2 ?

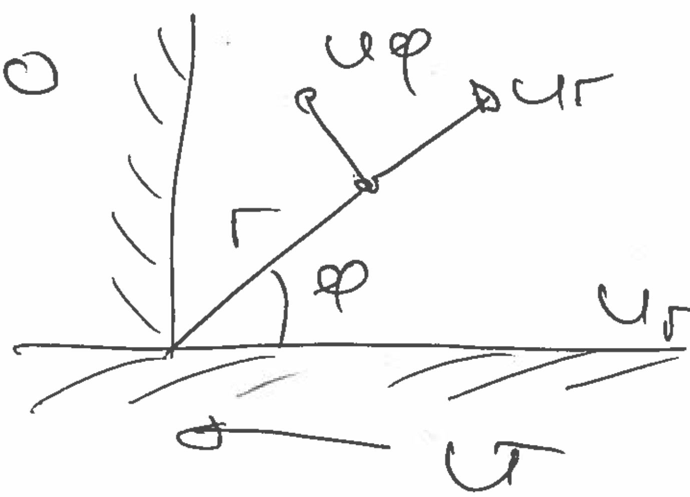
No fluid enters the corner $\Rightarrow$ streamfct has to be continuous across corner $\Rightarrow$ $C_1 = C_2 = C$

Choose $C = 0$ (on solid wall). (Adding a constant to ψ does not change $\underline{u}$).

$$\boxed{\begin{aligned} \psi(\varphi = 0) &= 0 \quad (1) \\ \psi\left(\varphi = \frac{\pi}{2}\right) &= 0 \quad (2) \end{aligned}}$$

<u>No slip:</u>

$u_r = u = 0$



$u_r = u = -U$

$$\boxed{u = 0 = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad \text{at } \varphi = \pi/2} \quad (3)$$

$$\boxed{u = -U = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad \text{at } \varphi = 0} \quad (4)$$

$$\nabla^4 \psi = \nabla^2 \nabla^2 \psi = 0$$

$$\psi(r, \varphi) = g(r) f(\varphi) \quad (\text{Ansatz!})$$

Note: (3) & (4): $\frac{1}{r}\frac{\partial \psi}{\partial \varphi}$ should be indep. of r (at fixed φ)

$\Rightarrow$ Try: $g(r) = Ur$

This allows (3) & (4) to be satisfied.

$$\psi(r, \varphi) = Ur f(\varphi)$$

BC for $f(\varphi)$:

$$\varphi = 0: \quad \psi = 0 \quad : \quad f(0) = 0$$

$$\varphi = \frac{\pi}{2}: \quad \psi = 0 \quad \quad f\left(\frac{\pi}{2}\right) = 0$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \varphi} = -U \text{ at } \varphi = 0 \quad : \quad f'(0) = -1$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \varphi} = 0 \text{ at } \varphi = \frac{\pi}{2} \quad : \quad f'\left(\frac{\pi}{2}\right) = 0$$

<u>PDE</u>: $\nabla^2 \nabla^2 \psi = 0$

$$\nabla^2 \psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) U r f(\varphi)$$

$$= 0 + \frac{U}{r} f + \frac{U}{r} f''$$

$$\nabla^2 \psi = U r^{-1} (f + f'')$$

$$\nabla^2 \nabla^2 \psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) U r^{-1} (f + f'')$$

$$= U \Big\{ (-1)(-2)\, r^{-3} \left(f + f'' \right) +$$

$$+ (-1)\, r^{-3} \left(f + f'' \right) +$$

$$+ r^{-3} \left(f'' + f^{IV} \right) \Big\}$$

$$= \frac{U}{r^3} \Big((2-1) f + (2-1+1) f'' + f^{IV} \Big)$$

$$\nabla^4 \psi = \frac{U}{r^3} \underbrace{\left(f + 2 f'' + f^{IV} \right)}_{\overset{!}{=} 0} \overset{!}{=} 0$$

$$f^{IV} + 2 f'' + f = 0$$

$$f(\varphi) \sim e^{\lambda \varphi}$$

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

$$(\lambda^2 + 1)^2 = 0$$

$\lambda_{1234} = \pm i$ repeated roots

$$f(\varphi) = A\sin(\varphi) + B\cos(\varphi) + C\varphi\sin(\varphi) + D\varphi\cos(\varphi).$$

Now apply 4BC $\Rightarrow$ A, B, C, D.

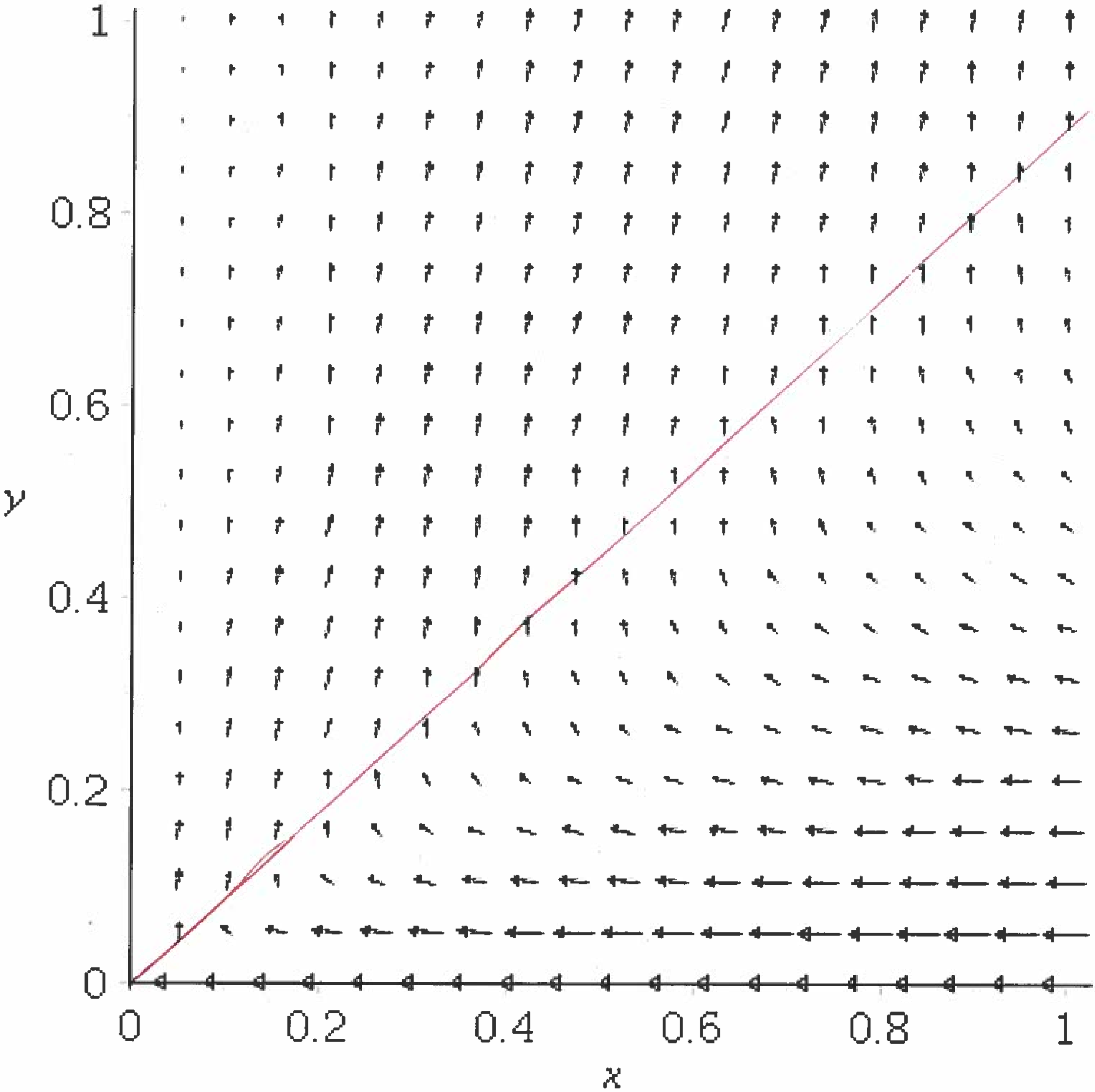
⋮

$$\psi(r,\varphi) = \frac{Ur}{\left(\frac{\pi}{2}\right)^2 - 1}\left(-\left(\frac{\pi}{2}\right)^2 \sin\varphi + \varphi\cos\varphi + \frac{\pi}{2}\varphi\sin\varphi \right)$$

Note:

$u_\varphi = v = -\frac{\partial\psi}{\partial r}$ does not depend on r.

$u_r = u = \frac{1}{r}\frac{\partial\psi}{\partial\varphi}$ ... either


1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x