Fluid mechanics

3 steps: I Describe (mathematically) the flow field/motion of fluid particles.

Kinematics

II Formulate the equations of motion (balance of forces on fluid particles; stress)

III Constitutive eqns. relate the kinematics to stresses.

} The Navier-Stokes eqns.

+ lots of examples!

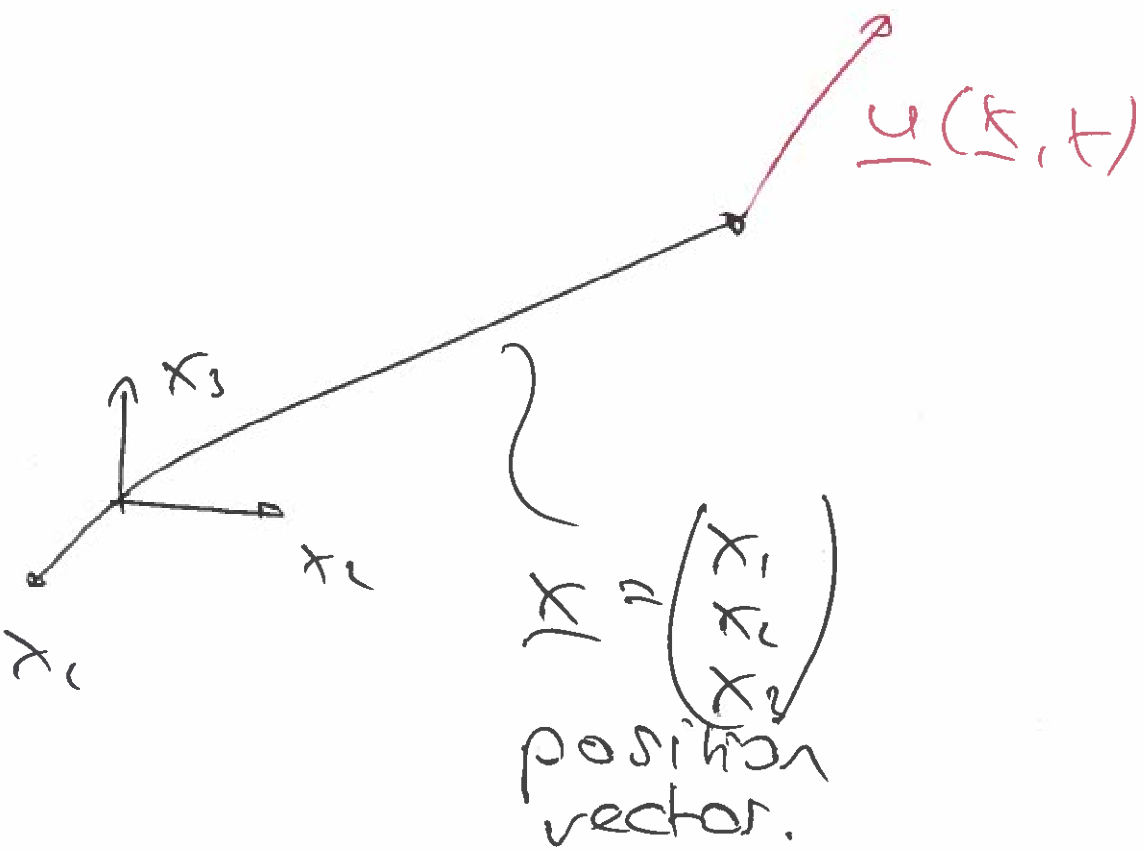
§ 2 Kinematics

The Eulerian flow field

Assume we know the velocity $\underline{u}$ as a fct. of the 3 cartesian coordinates $(x_1, x_2, x_3) = (x, y, z)$ & time t.

$$\underline{u} = \underline{u}(x_1, x_2, x_3, t) = \underline{u}(\underline{x}, t)$$

$$u_i = u_i(x_j, t)$$



<u>Note:</u> At different times different fluid particles occupy this position.

This has important implications, e.g. for the acceleration of fluid particles:

The material derivative

Follow particles: The position of a fluid particle is given by

$$\underline{x} = \underline{x}^p(t) = \begin{pmatrix} x_1^p(t) \\ x_2^p(t) \\ x_3^p(t) \end{pmatrix}$$

particle path (trajectory).

The velocity of this particle is given by

$$\underline{u} = \underline{u}\left(x_1^p(t), x_2^p(t), x_3^p(t), t\right)$$

So acceleration of the particle is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \left.\frac{\partial u}{\partial x_1}\right|_{\underline{x}^p} \frac{dx_1^p}{dt} + \left.\frac{\partial u}{\partial x_2}\right|_{\underline{x}^p} \frac{dx_2^p}{dt} + \left.\frac{\partial u}{\partial x_3}\right|_{\underline{x}^p} \frac{dx_3^p}{dt}$$

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_j} \frac{dx_j^p}{dt}$$

$$\frac{dx_j^p}{dt} = u_j$$

$$\boxed{\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}}$$

symbolically:

$$\boxed{\frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}}$$

often written as

$$\frac{D\underline{u}}{Dt} = \text{accel. of the particle that currently occupies } \underline{x}.$$

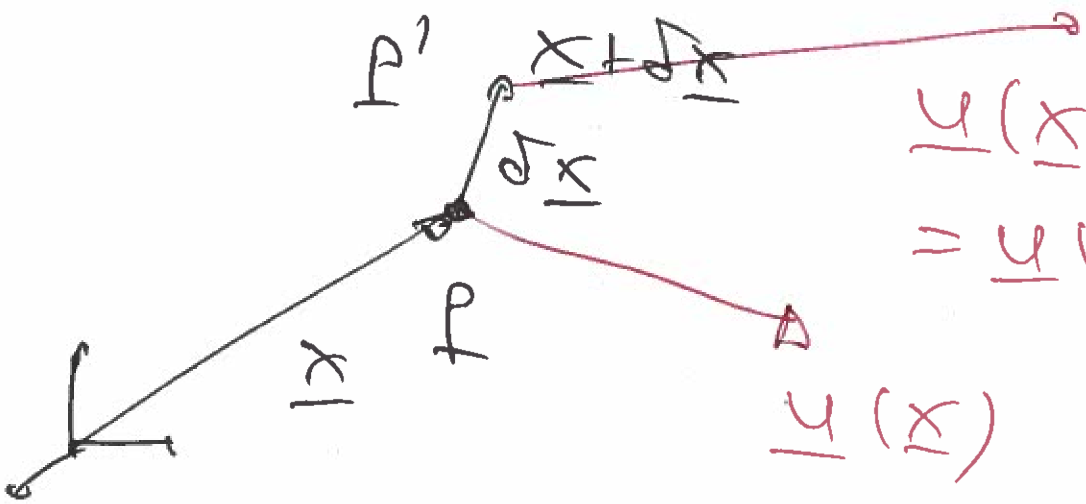
The rate of strain tensor & the vorticity

Velocity field itself is not very interesting.

It contains:
- translation
- rotation
- shearing
- dilation

How do we identify these?

Examine the veloc. field in the vicinity of a given point P.



$$\underline{u}(\underline{x}+\delta\underline{x}) = \underline{u}(\underline{x}) + \delta\underline{u}$$

The posn. changes from

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ to } \begin{pmatrix} x_1 + \delta x_1 \\ x_2 + \delta x_2 \\ x_3 + \delta x_3 \end{pmatrix}$$

$$\underline{u}(\underline{x} + \delta\underline{x}) = \underline{u}(x_1 + \delta x_1, x_2 + \delta x_2, x_3 + \delta x_3)$$

3d Taylor expansion:

$$= \underline{u}(x_1, x_2, x_3) + \frac{\partial \underline{u}}{\partial x_1}\delta x_1 + \frac{\partial \underline{u}}{\partial x_2}\delta x_2 + \frac{\partial \underline{u}}{\partial x_3}\delta x_3 + \dots$$

$\delta \underline{u}$

$\cancel{\delta u =}$

$$\delta u_i = \frac{\partial u_i}{\partial x_j}\,\delta x_j + \dots$$

Now $\delta x_i \to 0$

$$\boxed{du_i = \frac{\partial u_i}{\partial x_j}\,dx_j}$$

velocity gradient tensor (3x3 matrix)

If $\frac{\partial u_i}{\partial x_j} = 0$ then all fluid particles move with the same veloc.

$\Rightarrow$ translation.