3D:

$$\frac{D\underline{\omega}}{Dt} = \underbrace{\cancel{(\underline{\omega}\cdot\nabla)\underline{u}}}_{2D} + \nu\nabla^2\underline{\omega}$$

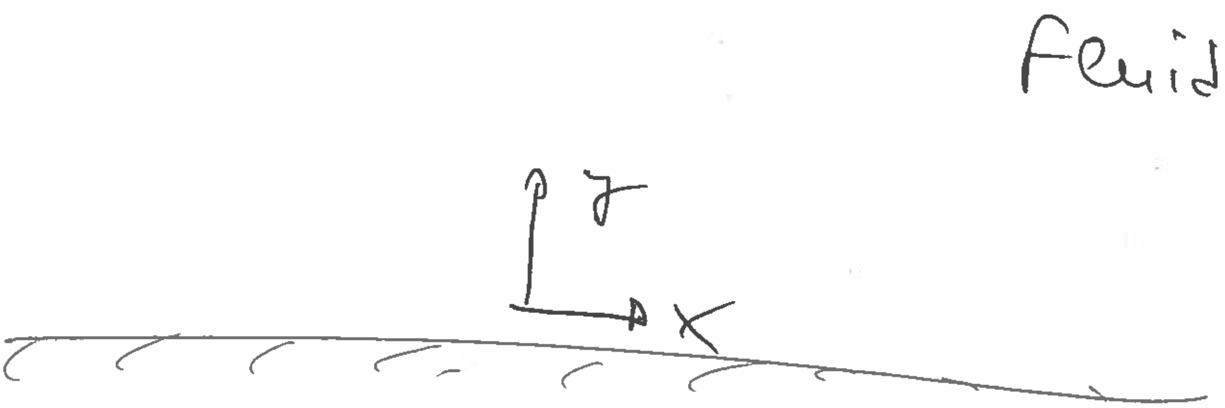
2D:

$$\underline{\omega} = \nabla\times\underline{u} = \omega\,\underline{e_z}$$

$$\omega = -\nabla^2\psi$$

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}$$

Example: flow due to a stretching plate



$$\underline{u} = Ux\,\underline{e_x}$$

Idea: find some solution to the PDEs & then adjust any constants to satisfy B.C.s.

Here: Example of a generalised Beltrami flow:

Ansatz:

$$\psi(x,y) = Ax\left(B - \exp(-\gamma y)\right)$$

A, B, γ adjustable constants.

$$\omega = -\nabla^2 \psi$$

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega$$

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

BC:

$$\left.\begin{aligned} u &= \frac{\partial \psi}{\partial y} = Ux \\ v &= -\frac{\partial \psi}{\partial x} = 0 \end{aligned}\right\} \text{ at } y = 0$$

$$u = A\gamma x \exp(-\gamma y)$$

$$u\big|_{y=0} = A\gamma x \overset{!}{=} Ux$$

$$\Rightarrow \underline{\underline{A = \frac{U}{\gamma}}}$$

$$v = -A(B - \exp(-\gamma y))$$

$$v\big|_{y=0} = -A(B-1) \overset{!}{=} 0 \Rightarrow \underline{\underline{B = 1}}$$

$$\psi = \frac{U}{\gamma} x (1 - \exp(-\gamma y))$$

$$u = Ux\exp(-\gamma y)$$

$$v = -\frac{U}{\gamma}(1 - \exp(-\gamma y))$$

$$\omega = -\nabla^2\psi = -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}$$

$$\frac{\partial^2\psi}{\partial x^2} = 0$$

$$\frac{\partial^2\psi}{\partial y^2} = -U\gamma x \exp(-\gamma y)$$

$$\omega = U\gamma x \exp(-\gamma y)$$

$$\frac{\partial\omega}{\partial x} = U\gamma \exp(-\gamma y)$$

$$\frac{\partial\omega}{\partial y} = -U\gamma^2 x \exp(-\gamma y)$$

$$\nabla^2\omega = +U\gamma^3 x \exp(-\gamma y)$$

into

$$\cancel{\frac{\partial\omega}{\partial t}} + u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = \nu\nabla^2\omega$$

$\boxed{5}$

$$\underbrace{Ux\exp(-\gamma y)}_{u}\;\underbrace{\cancel{U\gamma\exp(-\gamma y)}}_{\frac{\partial w}{\partial x}} +$$

$$+\underbrace{\left(-\frac{U}{\cancel{\gamma}}\right)\left(1-\exp(-\gamma y)\right)}_{v}\;\underbrace{\left(-\cancel{U}\cancel{\gamma}^{2}x\,\cancel{\exp(-\gamma y)}\right)}_{\frac{\partial w}{\partial y}}$$

$$=\underbrace{\cancel{U}\gamma^{\cancel{3}\,2}x\,\cancel{\exp(-\gamma y)}}_{\nabla^2 w}\,\nu$$

$$\cancel{Ux\exp(-\gamma y)}+Ux\left(1-\cancel{\exp(-\gamma y)}\right)=\nu\gamma^2 x$$

$$U=\nu\gamma^2$$

$$\gamma=\sqrt{\frac{U}{\nu}}$$

<u>So:</u>

$$u=Ux\exp\left(-\sqrt{\frac{U}{\nu}}\,y\right)$$

$$v=-\sqrt{U\nu}\left(1-\exp\left(-\sqrt{\frac{U}{\nu}}\,y\right)\right)$$

To sketch this as $y \to \infty$:

$u \to 0$

$v \to -\sqrt{UV}$

minus