§ 8 Stream function & vorticity equation

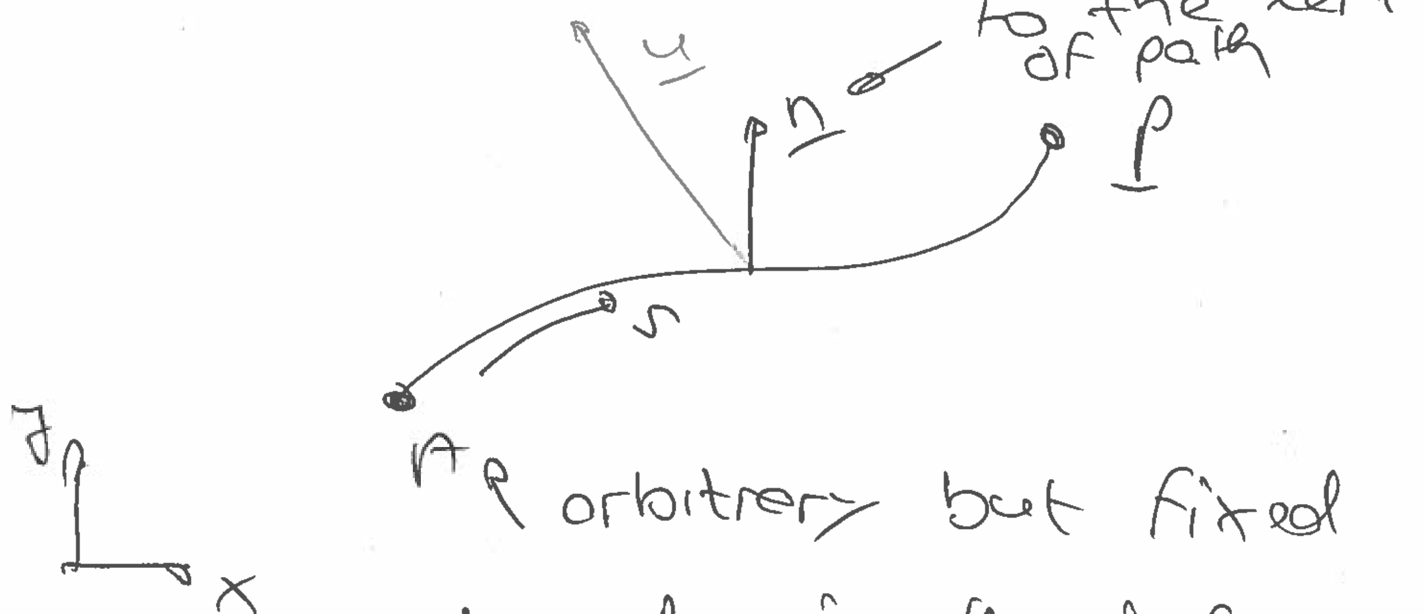
Alternative formulation of N. St. eqns, particularly useful in 2D.

Stream fct: (2D incompressible)

$$\underline{u} = u\,\underline{e_x} + v\,\underline{e_y}$$

Def:

$$\psi_A(P) = \int_A^P \underline{u} \cdot \underline{n}\, ds$$



A orbitrary but fixed

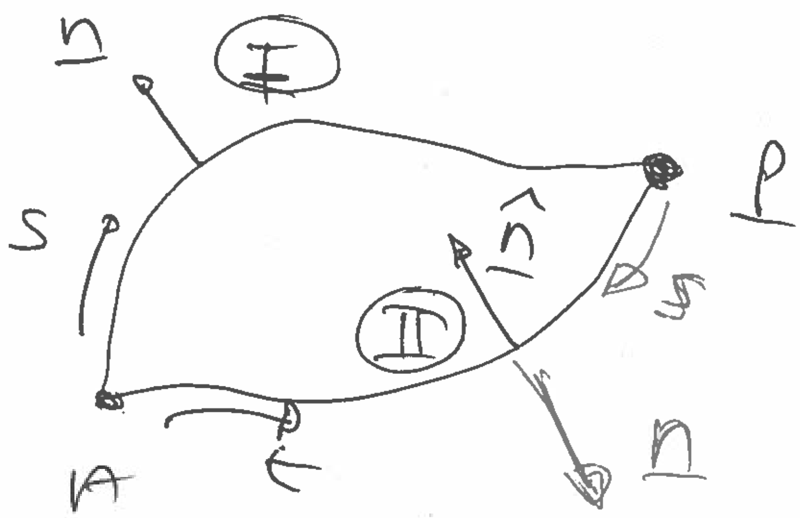
$\underline{u} \cdot \underline{n}\, ds$ is the infinitesimal volume flux (per unit depth)

$\Psi_A(P)$ = total volume flux (per unit depth) that crosses the line $A \to P$ (2

Implications:

(1) $\Psi_A(P)$ is path independent.

Proof:



II: $\underline{n} = -\hat{\underline{n}}$

$ds = -dt$

$$\Psi_A^{\mathrm{I}}(P) = \int_{A\,(\mathrm{I})}^{P} \underline{u}\cdot\underline{n}\,ds$$

$$\Psi_A^{\mathrm{II}}(P) = \int_{A\,(\mathrm{II})}^{P} \underline{u}\cdot\underbrace{\hat{\underline{n}}\,dt}_{-\underline{n}\quad -ds}$$

$$\Psi_A^{\mathrm{II}}(P) = -\int_{(\mathrm{II})}^{A} \underline{u}\cdot\underline{n}\,ds$$

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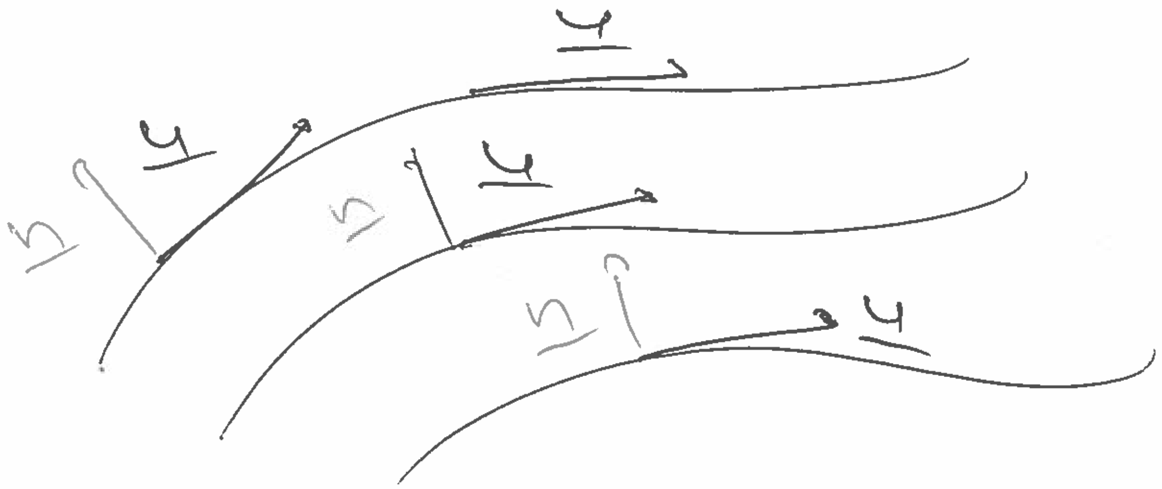
$$\psi_A^{\mathrm{I}}(P) - \psi_A^{\mathrm{II}}(P) = \int_{A\,(\mathrm{I})}^{P} \underline{u}\cdot\underline{n}\,ds + \int_{P\,(\mathrm{II})}^{A} \underline{u}\cdot\underline{n}\,ds$$

$$= \oint_{APA} \underline{u}\cdot\underline{n}\,ds = 0$$

(incompr.)

(2) ψ is constant along ~~stre~~ streamlines.
obvious since streamlines are tangential to the velocity field.



$\underline{u}\cdot\underline{n} = 0$ along streamlines!

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(3) impermeable solid boundaries are streamlines in the sense that

$\underline{u} \cdot \underline{n} = 0$ along them.

Convention: Set $\psi = 0$ along such boundaries.

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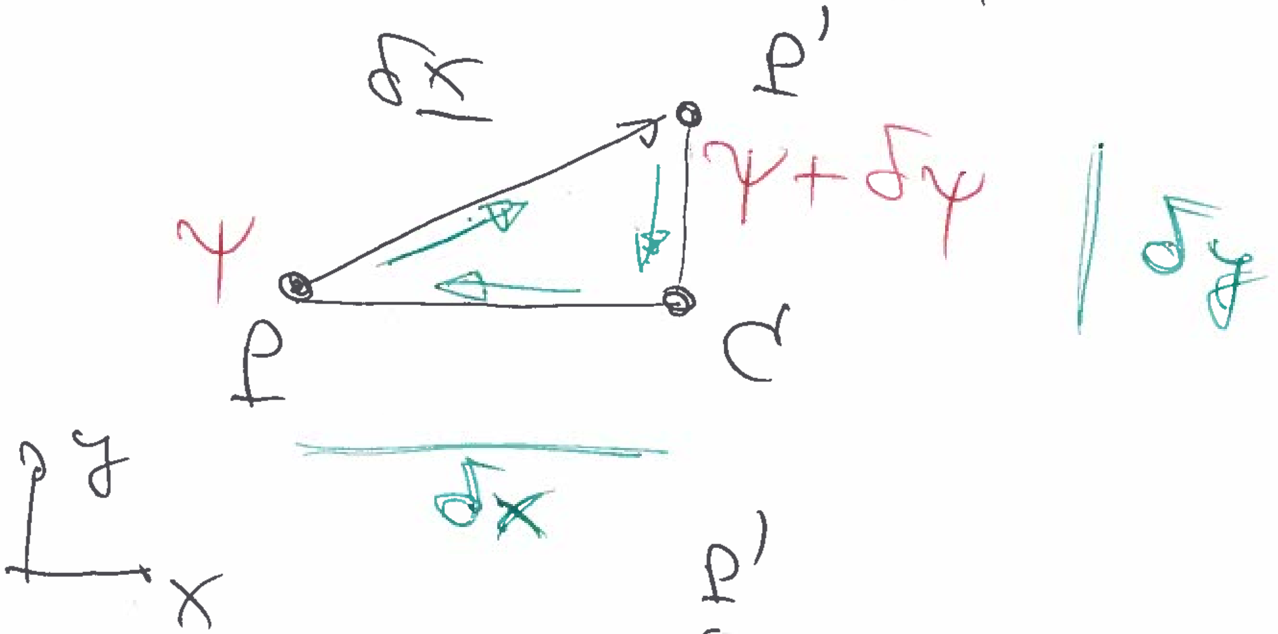
what PDE does ψ satisfy?



$$\delta\psi = \int_{P}^{P'} \underline{u} \cdot \underline{n} \, ds$$

use continuity:

$$\oint_{\triangle} \underline{u}\cdot\underline{n}\,ds = \underbrace{\int_P^{P'} \cdots}_{\delta\psi} + \int_{P'}^{C} \cdots + \int_{C}^{P} \cdots = 0$$

$$\delta\psi = \int_P^{P'} \underline{u}\cdot\underline{n}\,ds = -\int_{P'}^{C} \underbrace{\underline{u}}_{u}\cdot\underbrace{\underline{n}\,ds}_{(-dy)} - \int_{C}^{P} \underbrace{\underline{u}}_{(-v)}\cdot\underbrace{\underline{n}\,ds}_{(-dx)}$$

$$= \int_{P'}^{C} u\,dy - \int_{C}^{P} v\,dx$$

Now $\delta x, \delta y \to 0$

use the mean value theorem

$$\delta\psi = u\,\delta y - v\,\delta x$$

Also $\psi(x,y)$

expand:

$$\delta\psi = \frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y$$

$$\boxed{\begin{aligned}\frac{\partial\psi}{\partial x} &= -v\\ \frac{\partial\psi}{\partial y} &= u\end{aligned}}$$

Similar to potential, Airy stress fct. etc. Also similar to Cauchy Riemann..

Remarks:

(1) Derivation involved in continuity eqn.

$\Rightarrow$ This eqn. should already be satisfied.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \overset{?}{=} 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \overset{?}{=} 0 \quad \checkmark$$

(2) Relation to vorticity

$$\underline{\omega} = \nabla \times \underline{u} = \omega_z \underline{e_z} = \omega \underline{e_z}$$

2D!

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\omega = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$$

$$\boxed{\omega = -\nabla^2 \psi} \quad \text{in 2D.}$$