Similarity solns

(still §7)

$$u(y,t) = a(t)\, f\Big(\underbrace{\frac{y}{b(t)}}_{\eta}\Big)$$

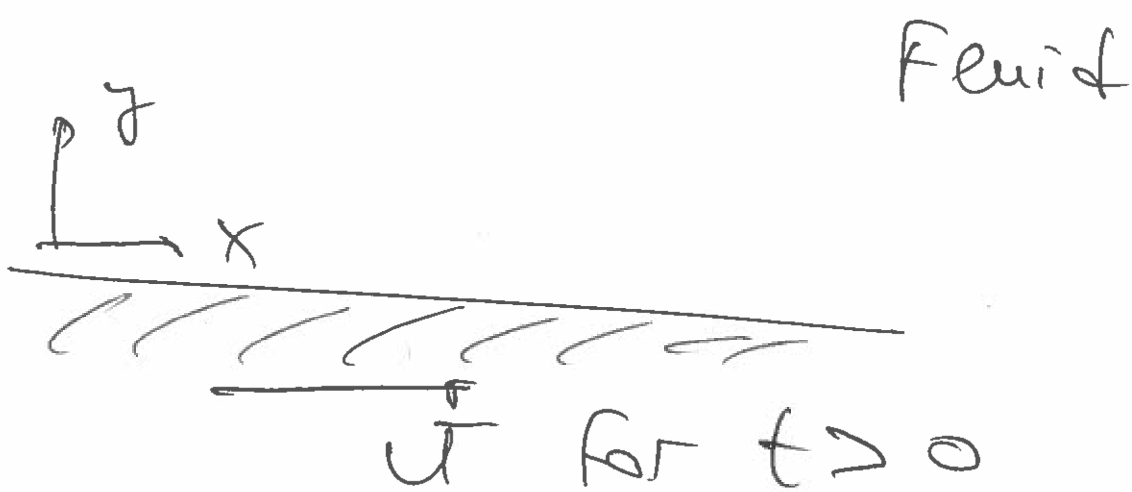
amplitude   shape   width

Existence of sim. solns. often suggested/implied by dimensional arguments/

Example:

Rayleigh's jerked plate:

IC: $u(y,t) = 0$ for $t \le 0$



Assume:

time-dep. parallel flow w/o body force & w/o press. gradients.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

BCs:

$$u\big|_{y=0} = U \quad \text{for } t > 0 \; (*)$$

$$u \to 0 \quad \text{as } y \to \infty$$

IC:

$$u(y, t=0) = 0$$

want $u(y, t; \nu, U)$

Note: PDE is linear & homog.
BC/IC are linear in u & homog. apart from (*)

$\Longrightarrow$ $u(y, t; \nu, U)$ must be linear in U !!

$$u(y,t;\nu,U) = U f(y,t;\nu)$$

Now check dimensions:

$$[u] = m/sec = [U]$$

$$[y] = m$$

$$[t] = sec$$

$$[\nu] = \frac{m^2}{sec}$$

$$\frac{\left[\frac{\partial u}{\partial t}\right]}{\left[\frac{\partial^2 u}{\partial y^2}\right]} = [\nu] = \frac{\cancel{m}}{sec^2}\,\frac{m^2\,\cancel{sec}}{\cancel{m}}$$

$$[\nu] = \frac{m^2}{sec}$$

$\Rightarrow$ f must be a dimensionless fct. of its arguments.

$$\underbrace{f(y,t;\nu)}$$

<u>must</u> combine

to a dimensionless combination:

$$\eta = \frac{t\nu}{y^2}$$

$$\eta = \frac{t^2\nu^2}{y^4}$$

or

$$\eta = \frac{y}{\sqrt{\nu t}}$$

choice is not unique.

$$u(y, t; \nu, U) = U f\Big(\underbrace{\frac{y}{\sqrt{\nu t}}}_{\eta(y,t)}\Big)$$

into eqns.:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\eta = \frac{y}{\sqrt{\nu}} t^{-1/2}$$

$$\frac{\partial u}{\partial t} = U \frac{df}{d\eta} \frac{\partial \eta}{\partial t}$$

$$\frac{\partial u}{\partial t} = U f' \left(-\frac{1}{2}\right) \frac{y}{\sqrt{\nu}} t^{-3/2}$$

$$\frac{\partial u}{\partial y} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial y}$$

$$= U f' \frac{1}{\sqrt{\nu t}}$$

$$\frac{\partial^2 u}{\partial y^2} = U f'' \frac{1}{\nu t}$$

into PDE:

$$\underbrace{\cancel{U} f' \left(-\frac{1}{2}\right) \frac{y}{\sqrt{\nu}\, \cancel{t}\sqrt{t}}}_{\frac{\partial u}{\partial t}} = \underbrace{\cancel{\nu} \cancel{U} f'' \frac{1}{\cancel{\nu t}}}_{\frac{\partial^2 u}{\partial y^2}}$$

η

$$\boxed{f'' + \frac{1}{2}\eta f' = 0}$$

ODE for $f(\eta)$

$$u(y,t) = U f(\eta)$$

$$\eta = \frac{y}{\sqrt{\nu t}}$$

BC:

$$u = U \quad \text{for } y = 0$$

$$\Downarrow$$

$$\eta = 0$$

$$\boxed{f(\eta = 0) = 1}$$

$$u \to 0 \quad \text{as} \quad y \to \infty$$

$$\boxed{f \to 0 \quad \text{as} \quad \eta \to \infty}$$

IC:

$$u = 0 \quad \text{as } t \to 0^+$$

$$\boxed{f \to 0 \quad \text{as} \quad \eta \to \infty}$$

(again!)

Solve:

$$f'' + \frac{1}{2}\eta f' = 0$$

$$F = f'$$

$$F' + \frac{1}{2}\eta F = 0$$

$$\frac{F'}{F} = -\frac{1}{2}\eta$$

$$\ln\left(\frac{F}{F_0}\right) = -\frac{1}{4}\eta^2$$

$$F = F_0 \exp\left(-\frac{1}{4}\eta^2\right) = f'$$

$$f(\eta) = A + F_0 \int^{\eta} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$

Lower limit? Arbitrary, just changes value of A

Choose $+\infty$

$$f(\eta) = A + f_0 \int_{\infty}^{\eta} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$

$$f(\eta) = A + B \int_{\eta}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$

<u>BC:</u>

$$f \to 0 \quad \text{as} \quad \eta \to \infty$$

$$\Rightarrow A = 0$$

$$f(\eta = 0) = 1 = B \underbrace{\int_{0}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi}_{\sqrt{\pi}}$$

$$u(y,t) = \frac{U}{\sqrt{\pi}} \int_{\eta = \frac{y}{\sqrt{\nu t}}}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$