(1

Example: Flow past sphere

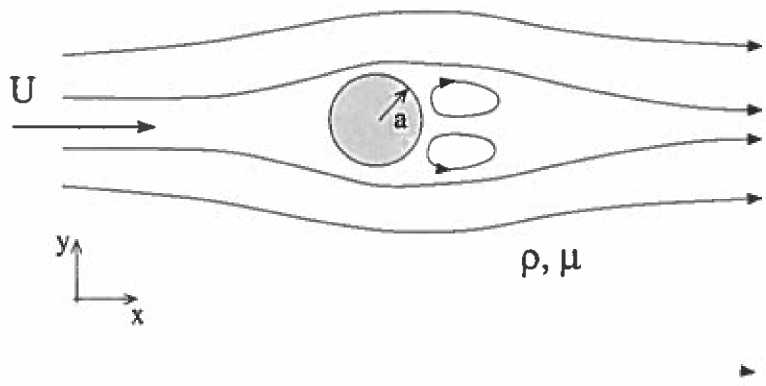


Figure 2: Flow past a sphere. Far away from the sphere of radius a, the fluid has a uniform velocity, $\mathbf{u} = U\mathbf{e}_x$.

- Scales:

  length scale: a

  velocity scale: U

  time scale: Steady boundary conditions, so there's no explicit time scale in the problem. Hence, we need to construct a time scale from the available parameters. Choose: $T = a/U$.

  pressure scale: There's no natural scale for the pressure. We can construct two reference pressures from the physical parameters.

  - $P = \rho U^2$ which is a dynamic pressure. This is appropriate if we expect dynamic effects to be dominant, i.e. for high velocity flows

  or

  - $P = \mu U/a$ which is a viscous pressure scale. This is appropriate if we expect viscous effects to be dominant, i.e. for slow flows with large viscosity.

(2

- Use these scales to non-dimensionalise the physical quantities:

$$u_i = U\tilde{u}_i$$

$$x_i = a\tilde{x}_i$$

$$t = \frac{a}{U}\tilde{t}$$

$$p = \begin{cases} \rho U^2 \tilde{p} & \text{for the dynamic pressure scale} \\ \mu U/a\ \tilde{p} & \text{for the viscous pressure scale} \end{cases}$$

- Inserting the scaled quantities into the Navier Stokes equations turns the problem of the flow past a sphere into

$$\begin{cases} Re\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \tilde{\nabla}^2 \tilde{u}_i & \text{for } p = \mu U/a\ \tilde{p} \\ \frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\tilde{\nabla}^2 \tilde{u}_i & \text{for } p = \rho U^2\ \tilde{p} \end{cases}$$

together with the continuity equation

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}_i} = 0,$$

and the boundary conditions

$\tilde{u}_i = 0$ for $\tilde{r} = 1$ (no slip on the surface of the sphere)

and

$\tilde{\mathbf{u}} \to \mathbf{e}_x$ as $\tilde{r} \to \infty$ (uniform velocity far away from the sphere).

- In non-dimensional form, the problem depends only on *one* dimensionless parameter, the *Reynolds number*

$$Re = \frac{\rho a U}{\mu} = \frac{aU}{\nu}$$

which represents the ratio of inertial to viscous forces in the flow.

$$\rho\left(\frac{U^2}{a}\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \frac{U^2}{a}\tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j}\right) = -\frac{\mu U}{a^2}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \mu\frac{U}{a^2}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}$$

$$u_i = U\tilde{u}_i$$

$$x_i = a\tilde{x}_i$$

$$t = \frac{a}{U}\tilde{t}$$

$$p = \frac{\mu U}{a}\tilde{p}$$

$$\frac{\rho U^{\cancel{2}}}{\cancel{a}}\frac{a^{\cancel{2}}}{\mu \cancel{U}}\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}$$

$$\boxed{\underbrace{\frac{\rho U a}{\mu}}_{Re}\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}}$$

↳ Reynolds number!

$$\frac{U}{a}\frac{\partial \tilde{u}_j}{\partial \tilde{x}_j} = 0$$

$$\boxed{\frac{\partial \tilde{u}_j}{\partial \tilde{x}_j} = 0}$$

No slip:

$$U\tilde{u}_i = 0 \quad \text{at } a\tilde{r} = a$$

$$\boxed{\tilde{u}_i = 0 \quad \text{at } \tilde{r} = 1}$$

Far field:

$$U\underline{\tilde{u}} \to U\underline{e_x} \quad \text{as } a\tilde{r} \to \infty$$

$$\boxed{\underline{\tilde{u}} \to \underline{e_x} \quad \text{as } \tilde{r} \to \infty}$$

Entire (non-dim.) solution only depends on one parameter: Re

- Hence, in non-dimensional terms, the solution will only depend on one parameter, i.e.

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}; Re).$$

- This implies that the non-dimensional velocity (the ratio of the actual velocity to the velocity far away from the sphere) at a fixed non-dimensional position (e.g. two diameters in front of the sphere) will have the same value for all physical realisations of the experiment provided the Reynolds number of the flows is the same.

- This means that an experiment with a 1:100 scale model of a jumbo jet will give the correct non-dimensional flow field, provided the velocity of the oncoming flow increased by a factor of 100 – and provided any physical effects which are not included in the incompressible Navier Stokes equations are unimportant. [The latter point is important in aerodynamics where compressibility often becomes an issue. Compressibility introduces another non-dimensional parameter (the Mach number) whose value also has to be conserved].

(6

Simplification of the scaled equations

- Writing the Navier Stokes equations in dimensionless form not only reduces the number free parameters, it also shows the appropriate limiting form of the equations if the Reynolds number approaches extreme values.
- For $Re \to 0$ (slow viscous flow), the viscous pressure scaling is appropriate. Performing the limit $Re \to 0$ in
$$Re\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \tilde{\nabla}^2 \tilde{u}_i$$
yields the *Stokes equations*:

$$Re = \frac{\rho U a}{\mu}$$

$$0 = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \tilde{\nabla}^2 \tilde{u}_i$$
which are linear since the non-linear inertial terms disappear.
- For $Re \to \infty$ (high speed flows), the inertial pressure scaling is appropriate. Performing the limit $Re \to \infty$ in
$$\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\tilde{\nabla}^2 \tilde{u}_i$$
shows that such flows are governed by the *Euler equations*
$$\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i}$$
- Note that the order of the Euler equations is lower than that of the full Navier Stokes equations (first rather than second spatial derivatives!). This means that not all boundary conditions can be applied on the surface of solid bodies.
- Typically, the no-slip condition is discarded in favour of the no-penetration condition (compare to inviscid flow theory which is also governed by these equations – in fact, the Euler equations can be derived by setting the viscosity to zero).

17

- However, close to the surface of the body, the no-slip condition always becomes important since viscosity (no matter how small) will always reduce the fluid velocity to zero as one approaches the surface of the solid body. This manifests itself in the existence of a thin layer (a so-called boundary layer) in which viscous effects are important and in which the velocity varies rapidly to fulfill the no-slip condition.
- Mathematically, the limit $Re \to \infty$ represents a *singular limit* and the solution has to be found by matched asymptotic expansions.
- We will briefly look at boundary layers at the end of this course.

Further comments

- The choice of the 'right' scales often requires some physical intuition. Especially when we use scaling arguments to simplify the equations (by dropping small terms), we have to choose the scales for the physical quantities such that the non-dimensional quantities are all of comparable magnitude ('of order one').

Flow past a circular cylinder (Panton p.387)

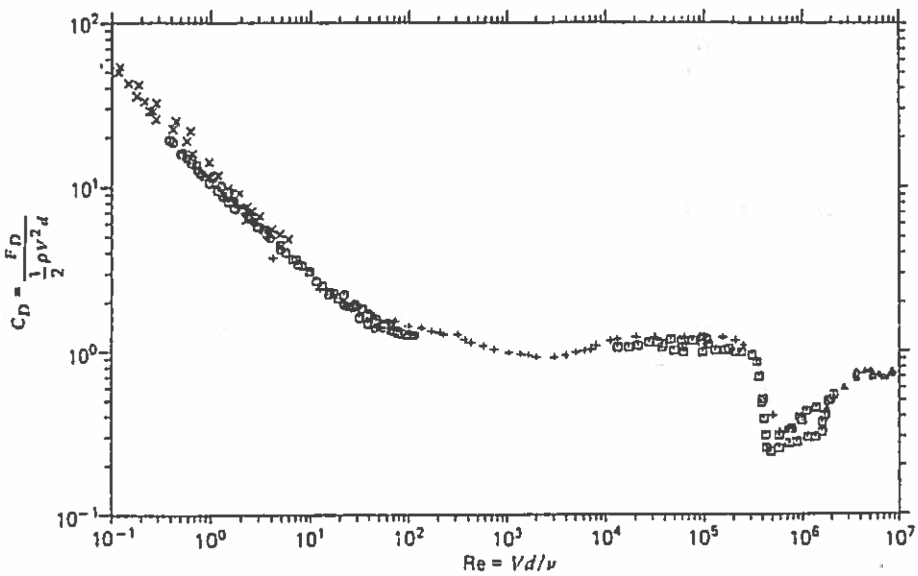
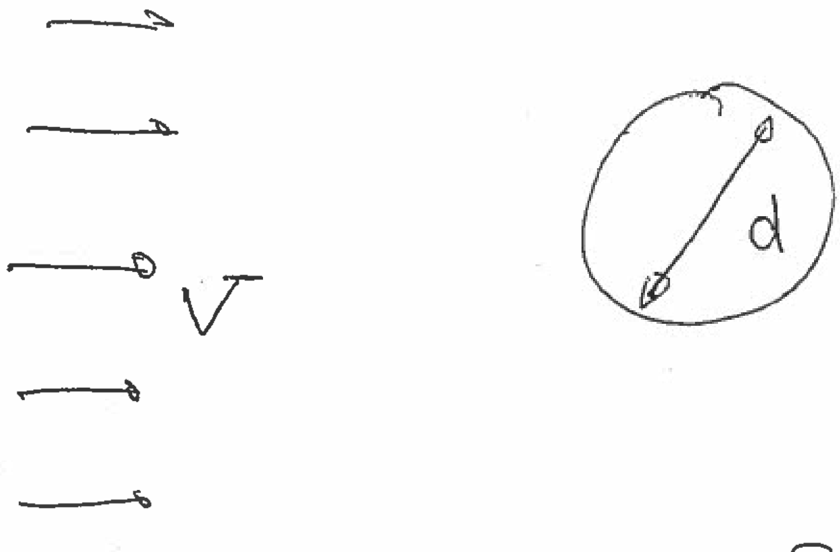


Figure 14.14 Drag curve for a cylinder. Data is from Delany and Sorenson (1953), Finn (1953), Roshko (1961), Tritton (1959), and Wieselsberger (1921).

Data from all kinds of exp. collapses onto the same curve i.e.

$C_D = f(Re)$



Drag force F_D

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 d} \qquad Re = \frac{Vd}{\nu}$$

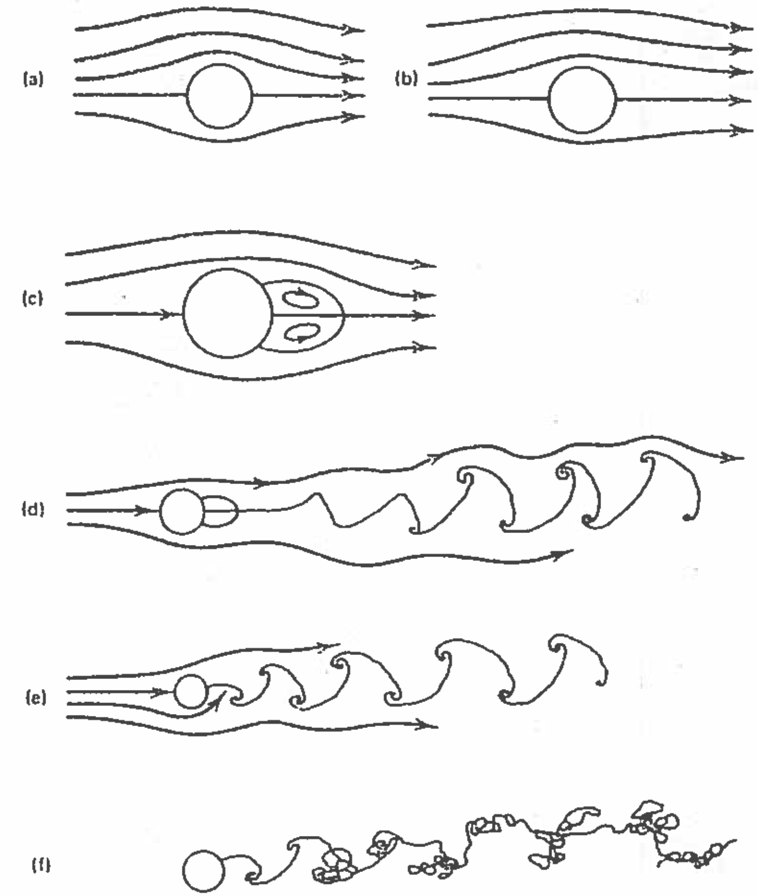


Figure 14.15 Flow regimes for a cylinder: (a) Re = 0, symmetrical; (b) 0 < Re < 4; (c) 4 < Re < 40, attached vortices; (d) 40 < Re < 60–100, von Kármán vortex street; (e) 60–100 < Re < 200, alternate shedding; (f) 200 < Re < 400, vortices unstable to spanwise bending.

then shed. Depending on the details of the experiment, this first occurs at a Reynolds number somewhere between 60 and 100. Figure 14.17a shows the vortex street development. As one goes further downstream the circular motion of the vortices is stopped by viscous forces. In an experiment such as Fig. 14.17 it is difficult to see when this happens as the flow visualization marker retains its distinctive pattern even after the vortices have stopped. The first picture in Fig. 14.18 shows a vortex pattern, the same as in Fig. 14.17, extending a distance of 200 diameters behind the (very small) cylinder. The path line streaks in the figure were produced by smoke from vaporizing oil on a hot wire located at the cylinder station. After some downstream

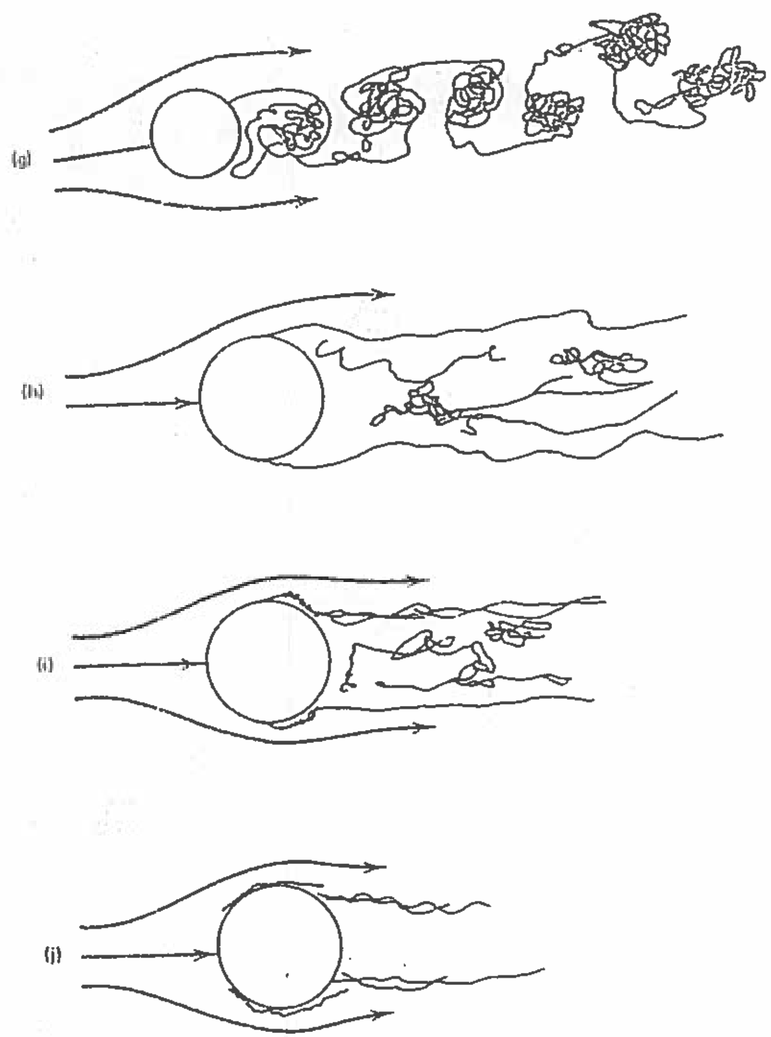


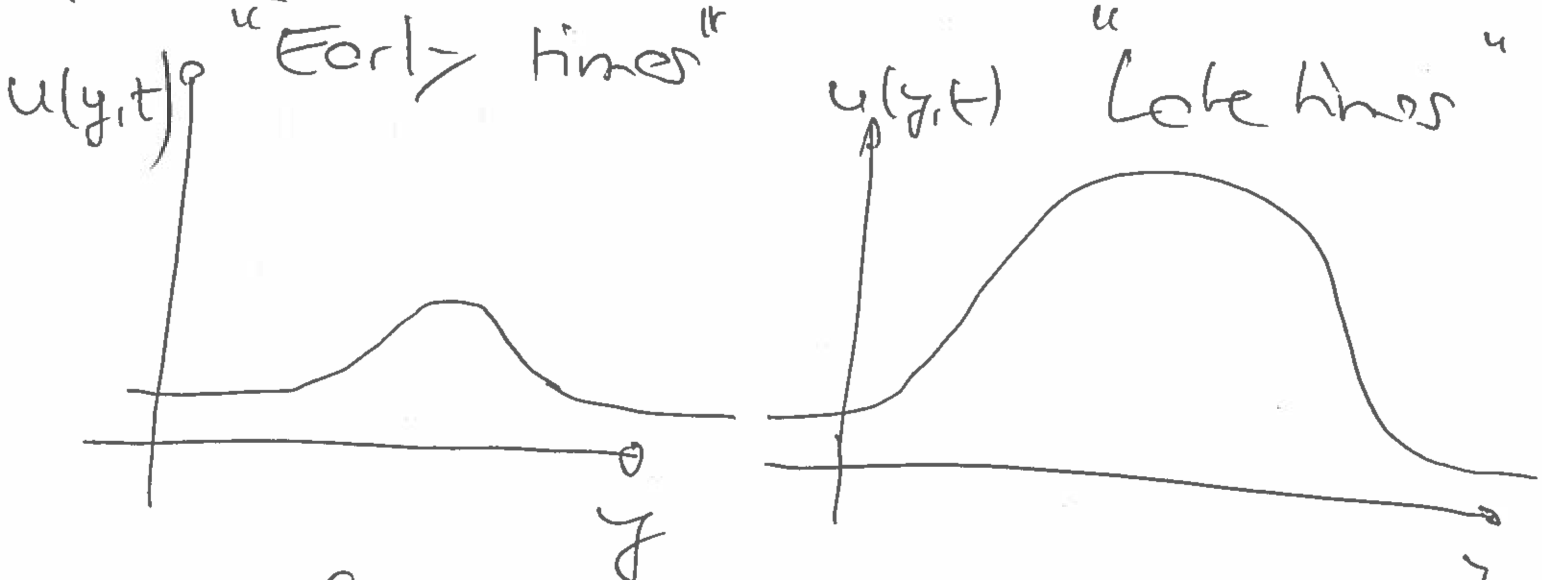
Figure 14.15 (*continued*) (g) 400 < Re, vortices turbulent at birth; (h) Re < 3×10^5, laminar boundary layer separates at 80°; (i) 3×10^5 < Re < 3×10^6, separated region becomes turbulent, reattaches, and separates again at 120°; (j) 3×10^6 < Re turbulent boundary layer begins on front and separates on back.

a location 150 diameters downstream from the cylinder and no vortices exist. The earlier patterns in the picture in Fig. 14.18a are fossils of events that occur where the smoke was introduced. Cimbala et al. (1988) have not only shown the vortex street decay, but they have also vividly demonstrated how

§7 Similarity solutions

Often solutions are "self-similar", i.e. they have the same shape but possibly different scales at different times.

E.g.:



This form of the solution is given by:

$$u(y,t) = \underbrace{a(t)}_{\text{amplitude}} f\left(\frac{y}{b(t)}\right)$$

$b(t)$ → scale for width

The existence of such self-similar solns. is often suggested by dimensional arguments.

TRY IT!

They often reduce a PDE for $u(y,t)$ to an ODE for $f(\eta)$ where $\eta = \dfrac{y}{b(t)}$ is the similarity variable.