Traction on fluid

$$u(r) = \frac{a^2\Omega}{b^2-a^2}\left(\frac{b^2}{r}-r\right)$$

$$\underline{n} = -\underline{e_r} = n_r\,\underline{e_r} + n_\varphi\,\underline{e_\varphi} + n_z\,\underline{e_z}$$

$$n_r = -1 \qquad n_\varphi = 0 \qquad n_z = 0$$

$$t_i = \tau_{ij} n_j \;; \quad \tau_{ij} = -p\,\delta_{ij} + 2\mu\,\varepsilon_{ij}$$

$$\varepsilon_{r\varphi} = -\frac{b^2\Omega}{b^2-a^2} \quad \text{at } r = a.$$

$$\Longrightarrow \boxed{t_r = p(r=a)}$$

$i = $ "φ":

$$t_\varphi = -\cancel{p\,n_\varphi} + 2\mu\big(\varepsilon_{\varphi r}\overset{-1}{n_r} + \cancel{\varepsilon_{\varphi\varphi} n_\varphi} + \cancel{\varepsilon_{\varphi z} n_z}\big)$$

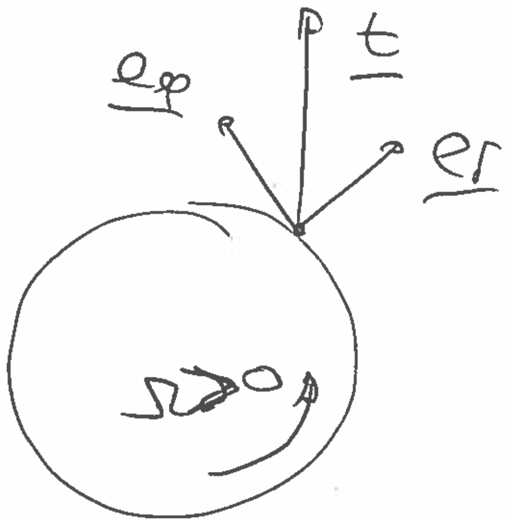
$$\boxed{t_\varphi = -2\mu\,\varepsilon_{r\varphi} = 2\mu\,\frac{b^2\Omega}{b^2-a^2}}$$

$$\boxed{t_z = 0}$$

$$\underline{t} = t_r \underline{e_r} + t_\varphi \underline{e_\varphi}$$

$$t_r = p(r=a)$$ Assume: > 0

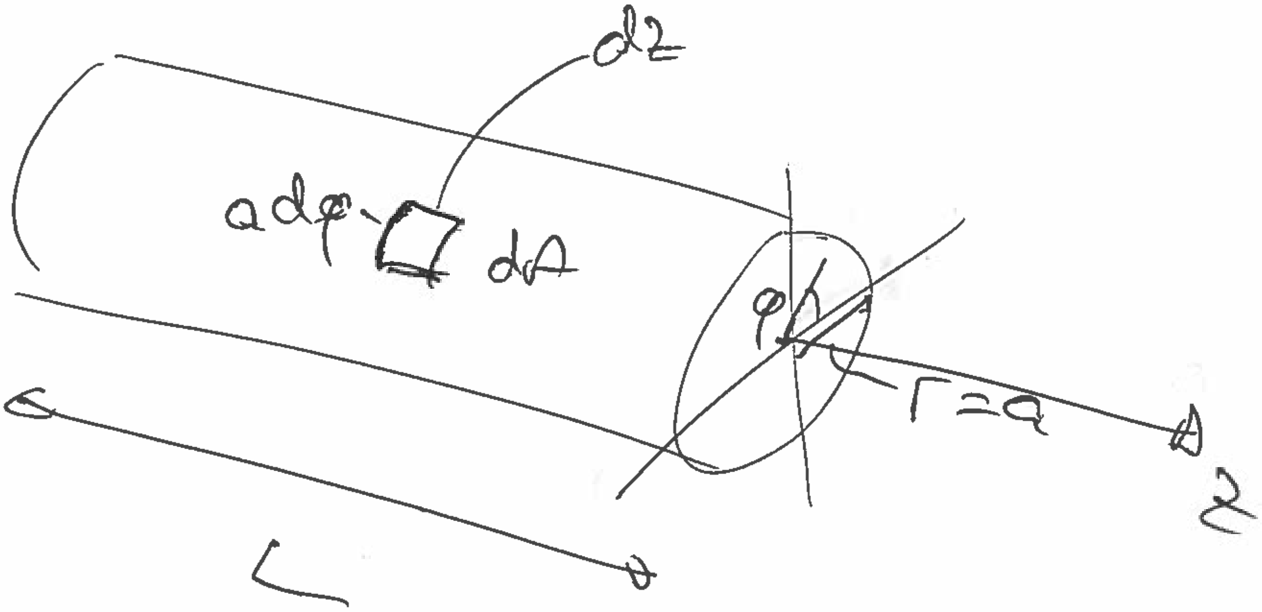
$$t_\varphi = 2\mu \frac{b^2 \Omega}{(b^2 - a^2)} > 0$$



$t_\varphi > 0$: because the fluid is dragged along by the rotating inner cylinder ($\Omega > 0$).

what about the resultant force? (on cylinder)

$$\underline{F}_{\substack{\text{onto}\\\text{cyl.}}} = \iint \underline{t}_{\substack{\text{onto}\\\text{cyl.}}}\, dA$$



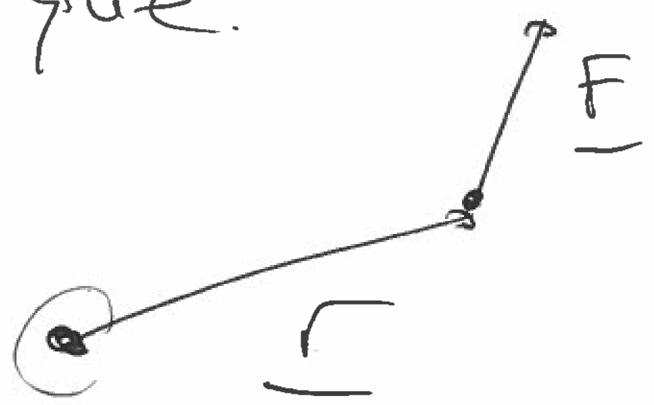
$$dA = a\, d\varphi\, dz$$

$$\underline{F}_{\substack{\text{onto}\\\text{cyl}}} = \int_{z=0}^{L} \int_{\varphi=0}^{2\pi} -(t_r \underline{e_r} + t_\varphi \underline{e_\varphi})\, a\, d\varphi\, dz$$

$$= La \int_0^{2\pi} -(\underbrace{t_r}_{\text{const}} \underline{e_r} + \underbrace{t_\varphi}_{\text{const}} \underline{e_\varphi})\, d\varphi = \underline{0}$$

No resultant force!

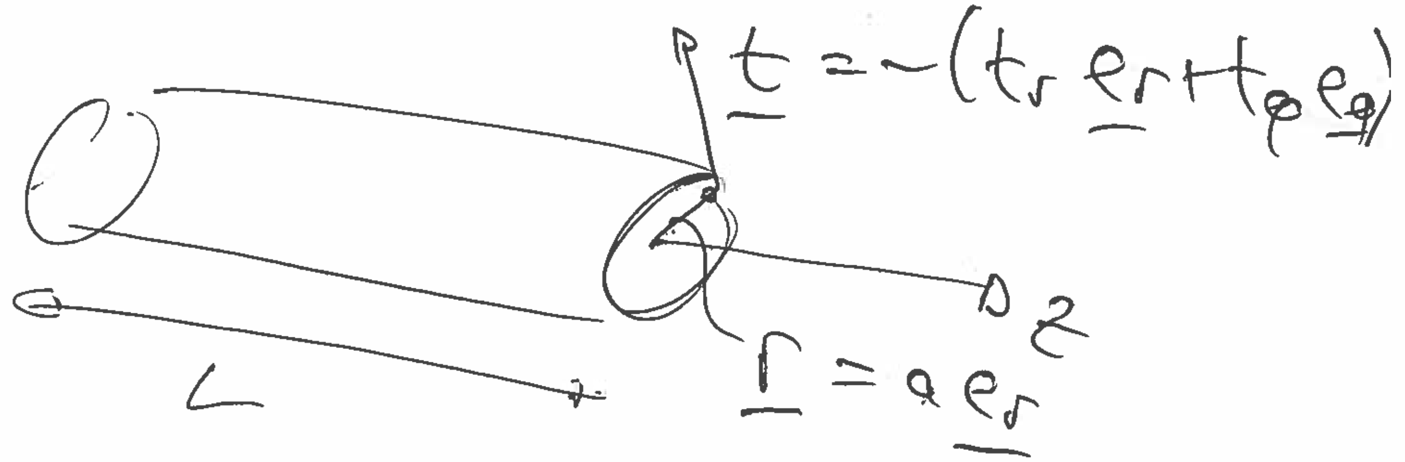
Cylinder is rotated by a torque.



$\underline{T} = \underline{r} \times \underline{F}$ for a point force $\underline{F}$

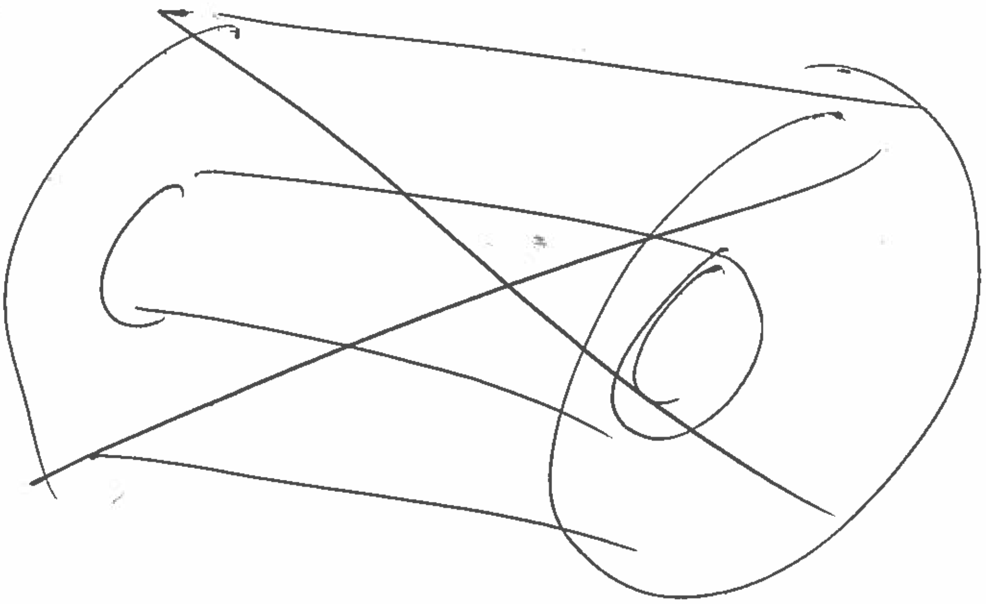
Generalise to traction

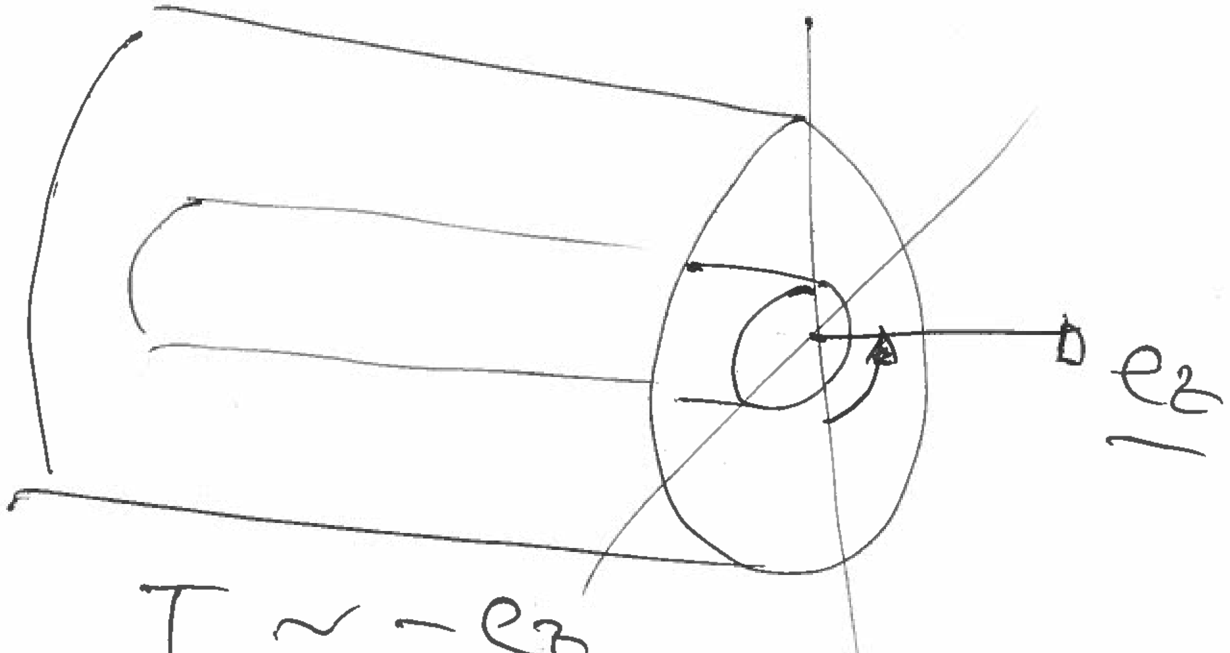
$$\underline{T} = \iint_{\text{wetted area}} -a\,\underline{e}_r \times \left(t_r \underline{e}_r + t_\varphi \underline{e}_\varphi\right) \underbrace{dA}_{a\,d\varphi\,dz}$$



$$\underline{T} = -L a^2 2\varpi t_\varphi \underline{e_z}$$

$$\boxed{\underline{T}_{\text{on cyl.}} = -4\pi \mu L \frac{a^2 b^2 \Omega}{b^2 - a^2} \underline{e_z}}$$





$$\underline{T} \sim -\underline{e_z}$$

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DIMENSIONAL ANALYSIS AND SCALING

Observation 1:

- Consider the flow past a sphere:

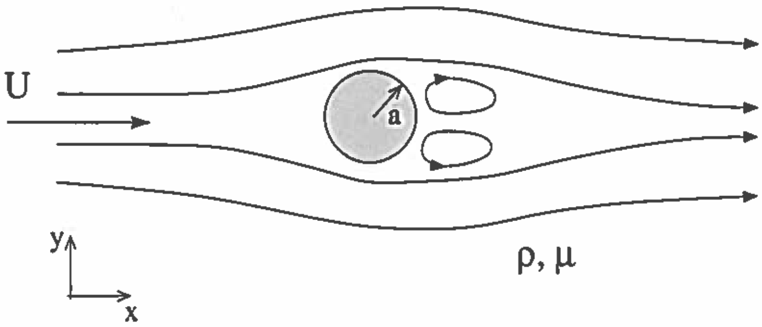

Figure 1: Flow past a sphere. Far away from the sphere of radius a, the fluid has a uniform velocity, $\mathbf{u} = U\mathbf{e}_x$.

- To determine the velocity field we need to solve the Navier-Stokes equations

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i$$

together with the continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0,$$

subject to the boundary conditions

$u_i = 0$ for $r = a$ (no slip on the surface of the sphere)

and

$\mathbf{u} \to U\mathbf{e}_x$ as $r \to \infty$ (uniform velocity far away from the sphere).

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- The solution which we could obtain by solving the equations analytically, numerically or even by carrying out experiments (!) will have the form

$$\mathbf{u} = \mathbf{u}(x, y, z, t; \rho, \mu, a, U),$$

i.e. the velocity field will depend on the spatial coordinates, on time and on the four physical parameters appearing in the problem.

- This implies that a change to any one of the physical parameters will (in general) change the entire flow field.
- This is not a problem if we can find an exact analytical solution which explicitly shows the dependence of the solution on each parameter.
- However, if we perform numerical computations (in which all parameters have to be given fixed numerical values), then each change of a physical quantity would require a completely new computation.
- If we perform experiments, then a different experiment has to be performed for each set of physical parameters (such as doubling the size of the sphere, making the fluid more viscous, etc.).
- Think of the implications for (e.g.) wind tunnel testing. If the above was true, then to obtain the flow field past a newly developed prototype car, you'd have to build the car in its full size. This might not be a problem but what about testing jumbo jets...?

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Observation 2:

- When we solve the Navier Stokes equations (or any other equation of continuum mechanics), we tend to get results like

$$u = \sin(r).$$

- Do we really?
- What about the dimensions of the above equation?

$$\underbrace{u}_{\text{m/sec}} = \sin(\underbrace{r}_{m}).$$

- How do you take the sin of 'metres'?
- Actually, we tend to get results like

$$u/U = \sin(r/a),$$

i.e. all quantities appear in dimensionless form.

- The fact that the equations of continuum mechanics are derived from (dimensionally coherent!) physical statements implies that we can *always* write our equations in dimensionless form.

Non-dimensionalisation

- We obtain non-dimensional equations by non-dimensionalising all quantities with characteristic scales which are in the problem. E.g.

$$\underbrace{\mathbf{u}}_{\text{dimensional velocity}} = \underbrace{U}_{\substack{\text{velocity scale:} \\ \text{velocity far from} \\ \text{the sphere}}} \underbrace{\tilde{\mathbf{u}}}_{\text{non-dimensional velocity}}$$

- Convention: Use a tilde to distinguish dimensional from non-dimensional variables (where necessary).
- The non-dimensionalisation typically reduces the number of free parameters in the problem and shows that 'similar' physical problems often have 'similar' solutions.
- A very useful side-effect of the non-dimensionalisation is that the non-dimensionalised equations provide additional insight into the relative size of the various terms in the equations (provided the 'scales' were chosen appropriately).
- The identification of small terms in an equation often motivates significant simplifications which can be obtained by neglecting the small terms against bigger ones.