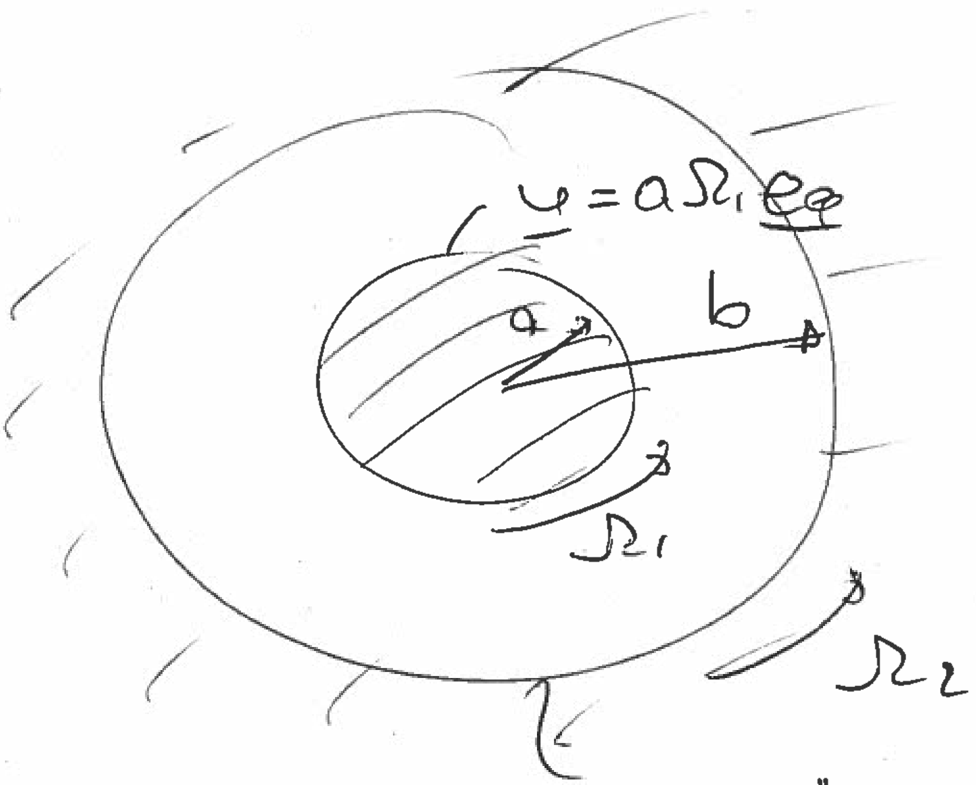


$$\underline{u} = b\Omega_2 \underline{e_\varphi}$$

Assumptions

- steady
- $\underline{u} = u_\varphi \underline{e_\varphi}$
- $\dfrac{\partial}{\partial \varphi} = 0$
- $\dfrac{\partial}{\partial z} = 0$

$$\left.\begin{array}{l} \text{steady} \\ \underline{u} = u_\varphi \underline{e_\varphi} \\ \dfrac{\partial}{\partial \varphi} = 0 \\ \dfrac{\partial}{\partial z} = 0 \end{array}\right\} \quad \begin{array}{l} \underline{u} = u_\varphi(r)\, \underline{e_\varphi} \\ \\ u_\varphi = U(r) \end{array}$$

As in plane Couette:

$$\nabla p = \underline{0}$$

Flow driven by walls.

Oh dear!

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} - \frac{v^2}{r} = \cancel{-\frac{1}{\rho}\frac{\partial P}{\partial r}} + \nu\left[\cancel{\nabla^2 u} - \cancel{\frac{u}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = \cancel{-\frac{1}{\rho r}\frac{\partial P}{\partial \varphi}} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \cancel{\frac{2}{r^2}\frac{\partial u}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial w}{\partial \varphi}} + \cancel{w\frac{\partial w}{\partial z}} = \cancel{-\frac{1}{\rho}\frac{\partial P}{\partial z}} + \cancel{\nu\nabla^2 w},$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}} + \cancel{\frac{\partial^2}{\partial z^2}}.$$

φ-momentum eqn:

$$\nabla^2 u - \frac{u}{r^2} = 0$$

$$\underbrace{\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}}_{\nabla^2 u(r)} - \frac{u}{r^2} = 0$$

$$\boxed{r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - u = 0}$$

2nd order ODE for $u(r)$

Euler ODE.

Ansatz: $u \sim r^\lambda$

into ODE:

$$r^\lambda \underbrace{\left(\lambda(\lambda-1) + \lambda - 1\right)}_{=0} = 0$$

$$\lambda^2 \cancel{-\lambda} \cancel{+\lambda} - 1 = 0$$

$$\lambda = \pm 1$$

$$v(r) = Ar + B\frac{1}{r}$$

A, B from BCs:

$$v(r=a) = a\Omega_1$$
$$v(r=b) = b\Omega_2$$

$$v(r) = \frac{1}{b^2-a^2}\left\{(b^2\Omega_2 - a^2\Omega_1)r - \frac{a^2b^2(\Omega_2-\Omega_1)}{r}\right\}$$

<u>Check:</u> $\Omega_1 = \Omega_2 = \Omega$

rigid body rotation?

$v(r) = \Omega r$ ✓

But the r-component of the mom. eqns is not satisfied!

Assumption that $\nabla p = \underline{0}$ is wrong!

Can fix this by assuming

$$p = p(r)$$

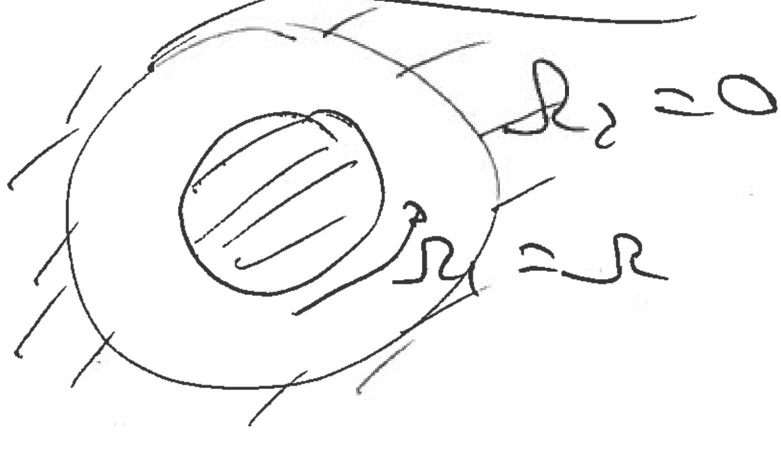
Into r-comp. of N-St:

$$-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{dp}{dr} = \rho \frac{v^2}{r}$$

where $v(r)$ is known so just integrate.

<u>Torque/moment/traction on inner cylinder</u>



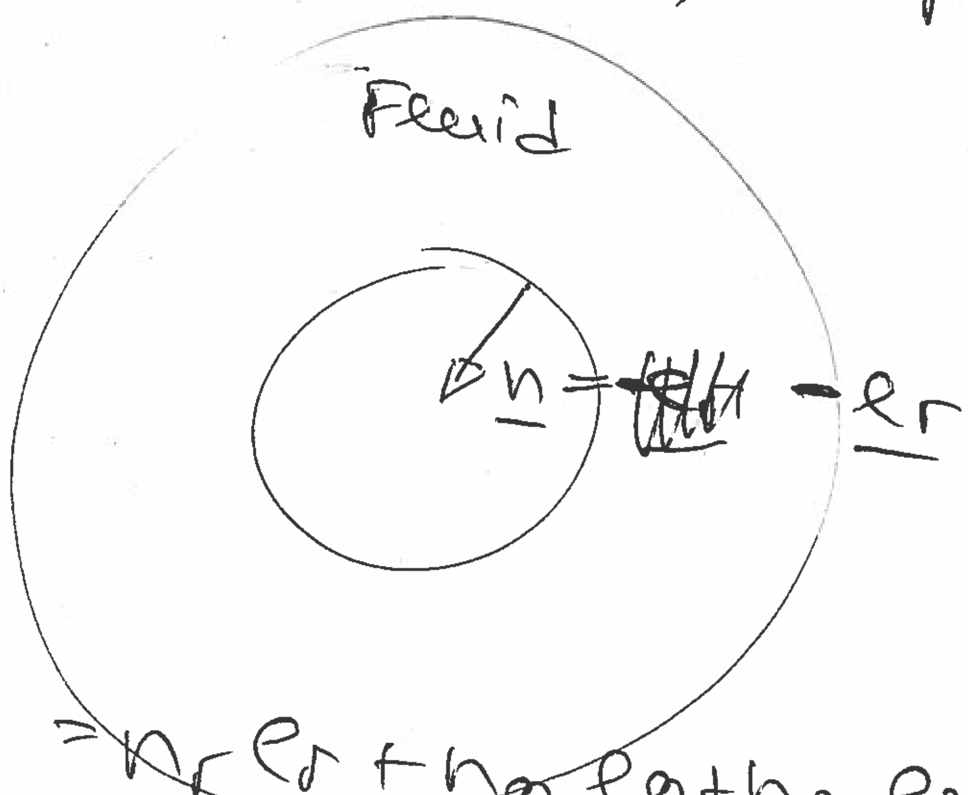
$$v(r) = \frac{a^2\Omega}{b^2-a^2}\left(\frac{b^2}{r} - r\right)$$

Traction from inner cylinder onto fluid:

$$\underline{t} = \underset{\sim}{T}\cdot\underline{n}$$

$$t_i = \tau_{ij} n_j \qquad i,j \text{ in } (r,\varphi,z)$$

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu\, e_{ij}$$



$$\underline{n} = n_r \underline{e}_r + n_\varphi \underline{e}_\varphi + n_z \underline{e}_z$$

$$n_r = -1 \qquad n_\varphi = n_z = 0$$

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \qquad \epsilon_{\varphi\varphi} = \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{u}{r}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} \qquad \epsilon_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \varphi}\right]$$

$$\epsilon_{\varphi z} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial w}{\partial \varphi} + \frac{\partial v}{\partial z}\right] \qquad \epsilon_{rz} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right]$$

Only nonzero entry in the rate of strain tensor is

$$\varepsilon_{r\varphi} = \frac{1}{2} r \frac{\partial}{\partial r}\left(\frac{v}{r}\right)$$

(see handout)

$$\varepsilon_{r\varphi} = -\frac{a^2 b^2 \Omega}{b^2 - a^2}\frac{1}{r^2}$$

On inner cylinder $r = a$:

$$\varepsilon_{r\varphi} = -\frac{b^2 \Omega}{b^2 - a^2}$$

$$t_i = -p n_i + 2\mu \varepsilon_{ij} n_j$$

$i = $ "r":

$$t_r = -p \underbrace{n_r}_{-1} + 2\mu\left[\cancel{\varepsilon_{rr}\underbrace{n_r}_{-1}} + \varepsilon_{r\varphi} \overset{0}{n_\varphi} + \cancel{\varepsilon_{rz} n_z}\right]$$

$$t_r = p(r = a)$$