§6 Curvilinear coordinates

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u}$$

$$\nabla\cdot\underline{u} = 0$$

change of coordinates also involves basis vectors

$$\underline{u} = u_x\underline{e_x} + u_y\underline{e_y} + u_z\underline{e_z}$$
$$= u_r\underline{e_r} + u_\varphi\underline{e_\varphi} + u_z\underline{e_z}$$

Basis vectors depend on the coordinates $\Rightarrow$ a mess

But can still use index notation for certain expressions

$$t_i = \tau_{ij} n_j \qquad \text{where } i,j = \{r,\varphi,z\}$$

$$\underline{t} = t_r\underline{e_r} + t_\varphi\underline{e_\varphi} + t_z\underline{e_z}$$

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu e_{ij}$$

↳ see handout

Illustration of one of the "additional" terms in cyl. polars.

Consider:

$$\underline{u} = u_\varphi \underline{e_\varphi}$$

$$u_\varphi = \Omega r = v$$

$$u_r = u_z = u = w = 0$$

in this case the radial component of the N.St. eqns becomes

$$-\rho \frac{v^2}{r} = -\frac{\partial p}{\partial r}$$

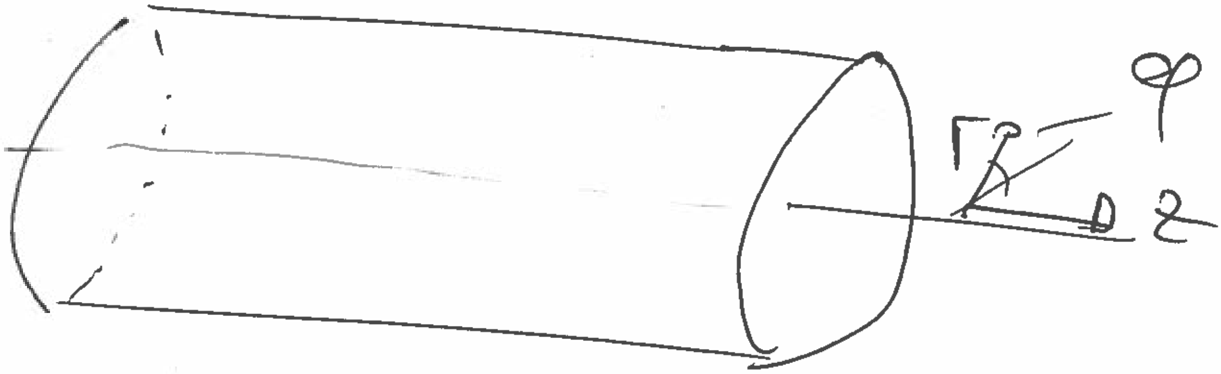
$$\rho \frac{\Omega^2 r^{\cancel{2}}}{\cancel{r}} = \frac{\partial p}{\partial r}$$

$$\frac{\partial p}{\partial r} = \rho \Omega^2 r \Rightarrow p(r) = p_0 + \frac{1}{2}\rho\Omega^2 r^2$$

centrifugal forces.

Example:

Hagen-Poiseuille flow



circular pipe, radius R
flow driven by pressure drop.

Assume: velocity is steady, indep. of φ, z & only in z-direction

$$\underline{u} = u_z(r)\,\underline{e_z} = w(r)\,\underline{e_z}$$

$$p = p(r,z) \quad \text{in fact } p = p(z)$$

z-comp:

$$0 = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\text{fct of } z} + \underbrace{\nu \nabla^2 w}_{\text{fct of } r}$$

$u = v = 0$ $\quad$ $\rho(\cancel{r}, z)$

$\omega(r)$

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} - \cancel{\frac{v^2}{r}} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\cancel{\nabla^2 u} - \cancel{\frac{u}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = -\cancel{\frac{1}{\rho r}\frac{\partial P}{\partial \varphi}} + \nu\left[\cancel{\nabla^2 v} - \cancel{\frac{v}{r^2}} + \cancel{\frac{2}{r^2}\frac{\partial u}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial w}{\partial \varphi}} + \cancel{w\frac{\partial w}{\partial z}} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w,$$

$$\operatorname{div}\mathbf{u} = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(ru)} + \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

where

$$\boxed{\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}} + \cancel{\frac{\partial^2}{\partial z^2}}.}$$

only consistent if

$$\frac{\partial p}{\partial z} = G = \text{const.}$$

$$\frac{G}{\rho} = \mu \nabla^2 w = \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r}\right)$$

$$\frac{G}{\mu} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r}\right)$$

$$\frac{G}{\mu} r = \frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r}\right)$$

$$\frac{1}{2}\frac{G}{\mu} r^2 + A = r \frac{\partial w}{\partial r}$$

$$\frac{1}{2}\frac{G}{\mu} r + \frac{A}{r} = \frac{\partial w}{\partial r}$$

$$w(r) = \frac{1}{4}\frac{G}{\mu} r^2 + A \ln(r) + B$$

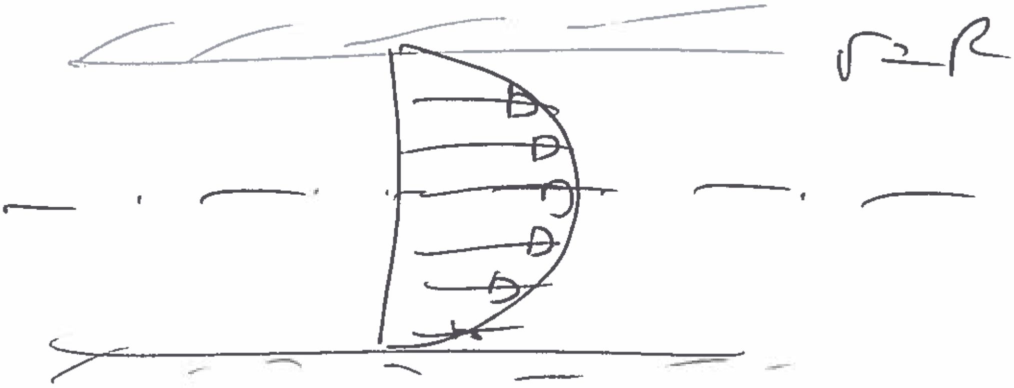
BC: No slip:

$w(r=R) = 0$

& veloc. finite at $r=0$

$\Rightarrow A = 0$

$$w(r) = \frac{1}{4}\frac{G}{\mu}\left(r^2 - R^2\right)$$



Example: Circular Couette flow