Parallel flow

$$\boxed{\begin{aligned} \rho \frac{\partial u}{\partial t} &= \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\ \rho f_y &= \frac{\partial p}{\partial y} \\ \rho f_z &= \frac{\partial p}{\partial z} \end{aligned}}$$

$$\underline{u} = u(y,z,t)\,\underline{e_x}$$

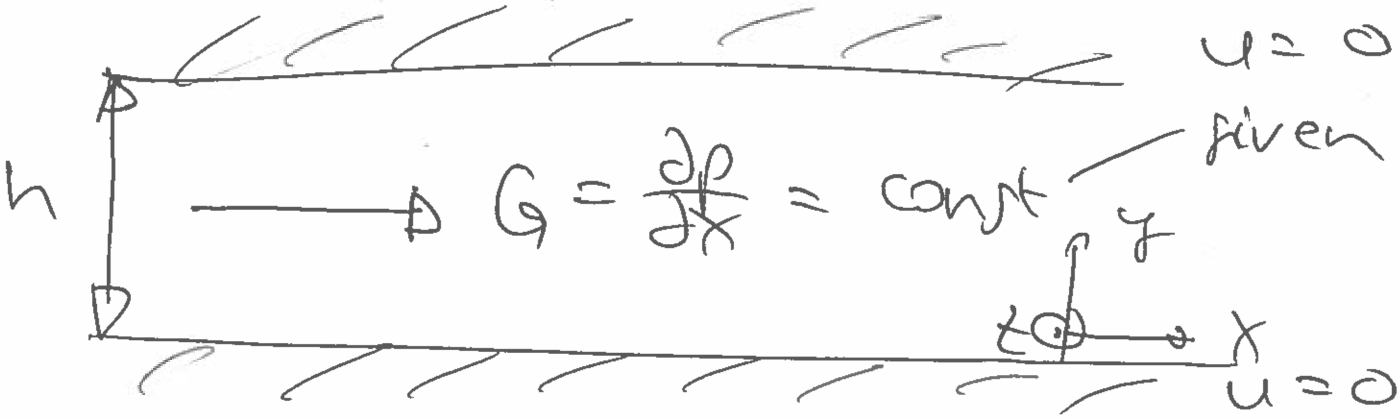
If $f = 0$:

$$\boxed{\begin{aligned} \rho \frac{\partial u}{\partial t} &= - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\ \frac{\partial p}{\partial x} &= G(t) \end{aligned}}$$

$$\nu = \mu/\rho$$

Example: Poiseuille flow

pressure-driven flow in a rigid channel



Assume:

$$u(y,z,t) = u(y)$$

$$\cancel{\rho \frac{\partial u}{\partial t}} = -\underbrace{\frac{\partial p}{\partial x}}_{G} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

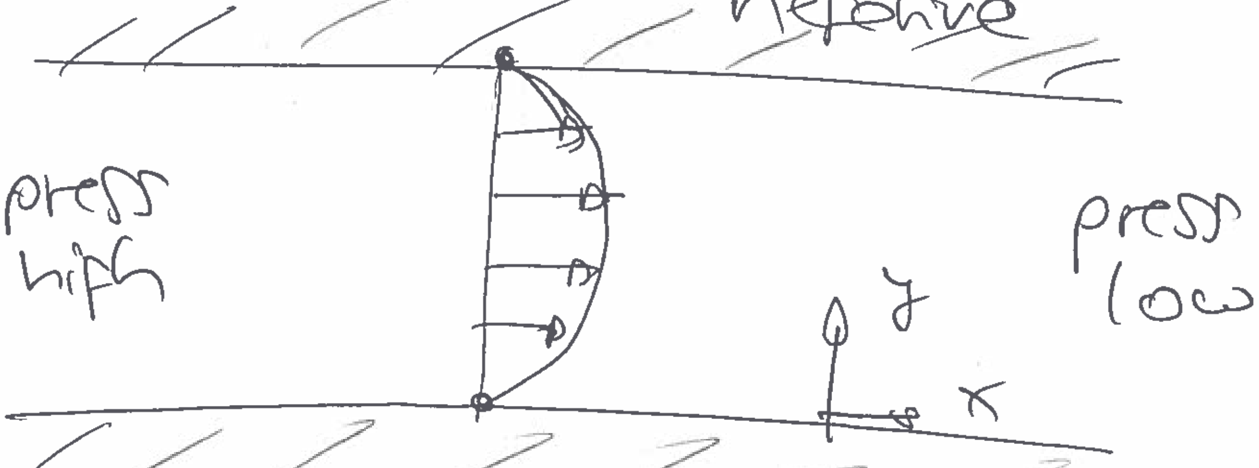
$$G = \mu \frac{d^2 u}{dy^2}$$ integrate twice

$$u(y) = \frac{1}{2}\frac{G}{\mu} y^2 + Ay + B$$

arb. const.

2BC: $u(y=0) = u(y=h) = 0$

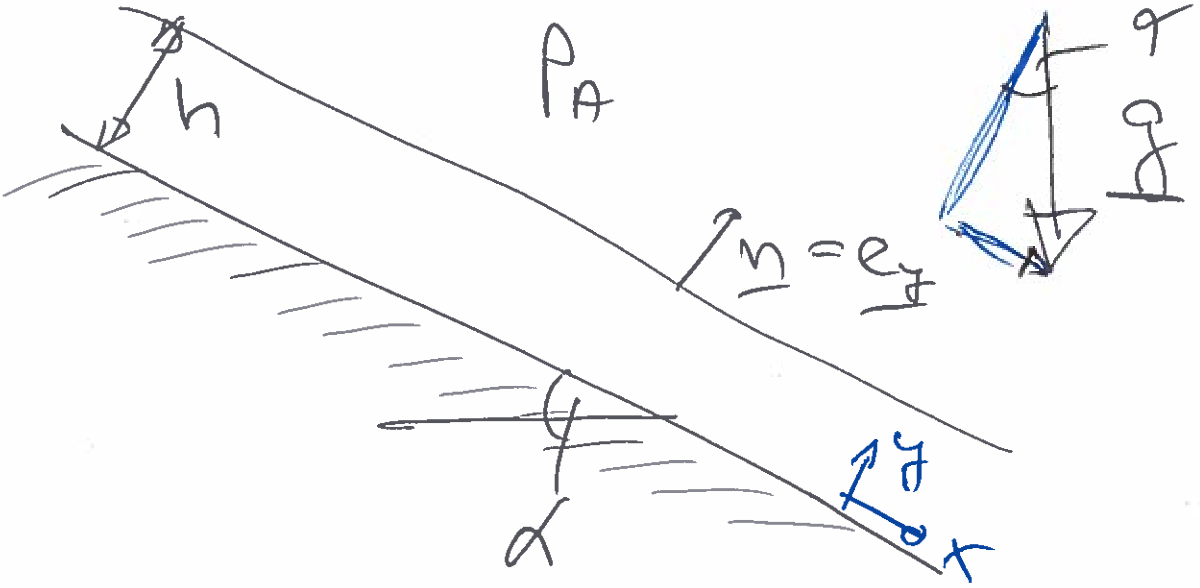
$$\boxed{u(y) = \frac{G}{2\mu}\underbrace{(y^2 - hy)}_{\text{negative}}}$$



if $G < 0$ recall $G = \frac{\partial p}{\partial x}$

Example:

Flow down an inclined plane



$$\underline{F} = \underline{g} = g \sin\alpha \, \underline{e}_x - g \cos\alpha \, \underline{e}_y$$

$$f_x = g \sin\alpha \; ; \; f_y = -g \cos\alpha$$

Assume: parallel flow, steady & indep. of z.

$$\cancel{\rho \frac{\partial u}{\partial t}} = \rho g \sin\alpha - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}} \right)$$

$$\boxed{0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin\alpha} \quad (1)$$

$$\boxed{0 = -\frac{\partial p}{\partial y} - \rho g \cos\alpha} \quad (2)$$

$$\boxed{0 = -\frac{\partial p}{\partial z}} \Rightarrow p \text{ does not depend on } z.$$

BC: ~~No~~ No-slip at $y=0$

$$\boxed{u(y=0)=0}$$

At $y=h$: free surface (steady)
Air (inviscid & at a pressure p_A) applies a traction onto the fluid:

$$\underline{t} = -p_A \underline{e_y}$$ = applied traction acting onto fluid.

$$t_1 = 0 \qquad t_2 = -p_A \qquad t_3 = 0$$

$\underline{n} = \underline{e_y}$

$n_1 = 0 \qquad n_2 = 1 \qquad n_3 = 0$

$t_i = \tau_{ij} n_j$

$t_i = \left[-p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] n_j$

at $y = h$

$t_i = -p n_i + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_j$

for $i = 1, 2, 3$

$\underline{i = 2}$:

$t_2 = -p_A = -p n_2 + \mu \left(\qquad \right) \cancel{n_1}$

$u_2 = 0 \rightarrow \quad + \mu \cancel{\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2} \right)} n_2$

$+ \mu \left(\qquad \right) \cancel{n_3}$

$n_2 = 1$

$$\boxed{p(y=h) = p_A}$$

<u>i=1:</u>

$$t_1 = 0 = -p\cancel{n_1} + \mu\left(\quad\right)\cancel{n_1}$$
$$+\mu\left(\frac{\partial u_1}{\partial x_2} + \cancel{\frac{\partial u_2}{\partial x_1}}\right)n_2$$
$$+\mu\left(\quad\right)\cancel{n_3}$$

$$0 = \mu\frac{\partial u_1}{\partial x_2}$$

$$\boxed{0 = \mu\frac{\partial u}{\partial y} \quad \text{at} \quad y=h}$$

no (tangential) shear stress.

<u>EXERCISE</u>

<u>i=3:</u> $\quad 0 = 0$

Integrate (2) w.r.t y

$$\frac{\partial p}{\partial y} = -\rho g \cos\alpha$$

$$p(x,y) = -\rho g \cos\alpha\, y + f(x)$$

BC: $p(y=h) = p_A$

$$\Rightarrow \boxed{p(x,y) = p_A + \rho g \cos\alpha\,(h-y)}$$

(~ hydrostatics)