# §1 Index notation

Various ways to write vectors:

$$\underline{a} = a_1 \underline{e_1} + a_2 \underline{e_2} + a_3 \underline{e_3} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

symbolic

in components relative to some basis $(\underline{e_1}, \underline{e_2}, \underline{e_3}) = (\underline{i}, \underline{j}, \underline{k})$

## Convention 1:

Simply write down one generic term of vector/vector eqn.

$$\underline{c} = \underline{a} + \underline{b}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

$c_i = a_i + b_i$

Free indices always represent all ~~terms~~ components of a vector (eqn.) : $i = 1, 2, 3$

$$\nabla \phi = \frac{\partial \phi}{\partial x_1} \underline{i} + \frac{\partial \phi}{\partial x_2} \underline{j} + \frac{\partial \phi}{\partial x_3} \underline{k}$$

$$= \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix} \Rightarrow \frac{\partial \phi}{\partial x_i}$$

Convention 2: Summation convention

Rule: Automatically sum over repeated indices (dummy indices)

E.g.: $\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

$$= \cancel{\sum_{i=1}^{3}} \underbrace{u_i v_i}$$

Name of index is irrelevant 3

$\underline{u} \cdot \underline{v} = u_i v_i = u_k v_k$

<u>Example:</u>

$\text{div}\, \underline{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$

$= \frac{\partial u_i}{\partial x_i}$

---

## Higher order "tensors"

So far: Index notation for vectors (3 components → one free index)
Higher-order tensors arise naturally in many applications

$\underline{\sigma} = \underset{\sim}{T} \cdot \underline{n}$

stress vector — stress tensor — normal vector

$$\sigma_i = T_{ij} n_j$$

(matrix-vector product)

---

## Kronecker Delta

A special $2^{nd}$ order tensor

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$[\delta_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

has an interesting property in summation:

$$b_j = a_i \delta_{ij} = \sum_{i=1}^{3} a_i \delta_{ij} = a_j$$

" $\delta_{ij}$ exchanges indices."