Chapter 1

Introduction

This set of notes summarises the main results of the lecture 'Viscous Fluid Flow' (MATH35001). Please email any corrections (yes, there might be the odd typo...) or suggestions for improvement to *M.Heil@maths.man.ac.uk*. Alternatively, see me after the lecture or in my office (Room 2.224 in the Alan Turing building).

Generally, the notes will be handed out after the material has been covered in the lecture. You can also download them from the WWW:

http://www.maths.man.ac.uk/˜mheil/Lectures/Fluids/.

This WWW page will also contain announcements, example sheets, solutions, etc.

1.1 Literature

The following is a list of books that I found useful in preparing this lecture. **It is not necessary to purchase any of these books!** Your lecture notes and these handouts will be completely sufficient.

Acheson, D.J. 1990 Elementary Fluid Dynamics. Clarendon Press, Oxford, 1990.

Spiegel, M. 1974 Vector Calculus. McGraw Hill (Schaum's Outline series).

Batchelor, G.K. 1967 An Introduction to Fluid Dynamics. Cambridge.

Sherman, F.S. 1990 Viscous Flow. McGraw Hill.

McCormack , P.S. & Crane, L.J. 1973 Physical Fluid Dynamics, Academic Press.

Panton, R.L. 1996 Incompressible Flow (second edition), Wiley.

White, F.M. 1991 Viscous Fluid Flow (second edition), McGraw Hill.

1.2 Preliminaries: Index notation & summation convention

- We will denote vectors/matrices/tensors by their components relative to a set of basis vectors. E.g. instead of writing $\mathbf{r} = r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 = (r_1, r_2, r_3)$, we simply write r_i and use the convention that all 'free indices' (i in this case) range from 1 to 3. Similarly, we represent the matrix
$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$
by its generic component a_{ij} and imply that the two free indices (i and j) take on all values in the range from 1 to 3.

- In general, we only write down the generic term of any vector (or vector equation). For instance $a_i = b_i + c_i$ is taken to represent the three components $a_1 = b_1 + c_1, a_2 = b_2 + c_2, a_3 = b_3 + c_3$ of the symbolic vector equation $\mathbf{a} = \mathbf{b} + \mathbf{c}$.

- Consistency check: Every term in an equation in index notation has to have the same number of 'free indices'. For instance, the addition of two matrices can be expressed as $A_{ij} = B_{ij} + C_{ij}$, whereas the equation $A_{ij} = B_{ik} + C_{lm}$ does not make sense.

- Kronecker Delta: $\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$

- Summation convention: Automatic summation over repated indices. Examples are:

  Dot product: $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{3} a_i b_i$. Sums like this will occur very frequently and it will turn out to be convenient to drop the summation sign and to automatically sum over any repeated index. I.e. $\sum_{i=1}^{3} a_i b_i = a_i b_i = a_k b_k$. Note that the 'name' of the summation index is irrelevant as it does not appear in the final result; therefore $a_i b_i$ is the same as $a_k b_k$. Summation indices are often called 'dummy indices'.

  Matrix-vector products: $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ becomes $A_{ij} x_j$ (or $A_{im} x_m = b_i$, say). Similarly $\mathbf{A}^T \cdot \mathbf{x} = \mathbf{c}$ becomes $A_{ji} x_j = c_i$ (or $A_{jk} x_j = c_k$, say). Note that the result of the matrix-vector product is a vector: Hence both sides of the equations have one (matching!) free index.

  δ_{ij} 'exchanges' indices: $a_i \delta_{ij} = a_j$.

- Comma denotes partial differentiation: E.g. $\frac{\partial u_i}{\partial x_j} = u_{i,j}$.

- Some differential operators in index notation:

$$\nabla \cdot \mathbf{u} = \text{div } \mathbf{u} = u_{i,i} \tag{1.1}$$

$$\nabla \phi = \text{grad} \phi = \phi_{,i} \tag{1.2}$$

$$\nabla^2 \phi = \phi_{,ii} \tag{1.3}$$

Chapter 2

The Kinematics of Fluid Flow

2.1 The Eulerian flow field

- Eulerian description of the flow field: The velocity $\mathbf{u}$ is given as a function of the position relative to a spatially fixed coordinate system $(x, y, z) = (x_1, x_2, x_3) = x_i$, and of time t.

$$\mathbf{u} = \mathbf{u}(x, y, z, t) \quad \text{or in index notation:} \quad u_i = u_i(x_j, t). \tag{2.1}$$

- Note that at different times, different material particles will be at a given spatial position. The particle paths (i.e. the trajectories $x_i^p(t)$ of individual material particles which are at position $x_i^{(0)}$ at time $t = t_0$) are obtained by integrating

$$\frac{\partial x_i^p(t)}{\partial t} = u_i(x_j^p, t) \tag{2.2}$$

subject to the initial conditions

$$x_i^p(t = 0) = x_i^{(0)}. \tag{2.3}$$

2.2 The material derivative

- The acceleration a_i of the material particle that is at position x_j at time t is given by

$$a_i(x_j, t) = \left(\frac{d}{dt} \; u_i(x_j^p(t), t) \right) \bigg|_{x_j^p(t) = x_j} = \frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_k} \frac{\partial x_k^p(t)}{\partial t}. \tag{2.4}$$

Comparing this to (2.2) shows that this can be written as

$$a_i = \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} \quad \text{or symbolically} \quad \mathbf{a} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{2.5}$$

- The differential operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k} \quad \text{or symbolically} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \tag{2.6}$$

is known as the 'material (or substantial) derivative'. Given any function $\phi(x_j, t)$, $D\phi/Dt$ represents the rate of change of ϕ experienced by an observer travelling with the velocity $u_i(x_j, t)$.

2.3 Vorticity and the rate of strain tensor

- The velocity field can be decomposed into four fundamental 'modes' which correspond to the translation, rotation, shearing and dilation of small material elements contained in the flow. The velocity in the vicinity of a certain point x_k can be expressed as

$$u_i(x_k + \delta x_k) = \underbrace{u_i(x_k)}_{\text{rigid body translation}} + \underbrace{\omega_{ij} \; \delta x_j}_{\text{rigid body rotation}} + \underbrace{\epsilon_{ij} \; \delta x_j}_{\text{shearing and dilation}} , \tag{2.7}$$

where ω_{ij} is the antisymmetric rate of rotation tensor

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{2.8}$$

and ϵ_{ij} is the symmetric rate of strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.9}$$

- The first term in (2.7) represents a rigid body translation: If $\epsilon_{ij} = \omega_{ij} = 0$ then all particles have the same velocity, i.e. the fluid moves in a straight line as a rigid body.
- The physical meaning of the second term in (2.7) is revealed by rewriting $\omega_{ij}\ \delta x_j$ symbolically as a cross product in the form $\mathbf{\Omega} \times \boldsymbol{\delta x}$ where $\mathbf{\Omega} = (\omega_{32}, \omega_{13}, \omega_{21})$ is the rate of rotation vector.

  This is illustrated in Fig. 2.1: The differential velocity $\delta\mathbf{u} = \mathbf{u}(x_j) - \mathbf{u}(x_j + \delta x_j)$ induced by a rigid body rotation about point P with rotation rate $\mathbf{\Omega}$ is given by $\delta\mathbf{u} = \mathbf{\Omega} \times \boldsymbol{\delta x}$.

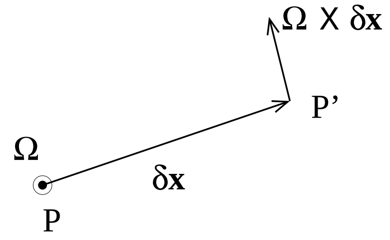

Figure 2.1: Sketch illustrating the motion induced by a rigid body rotation about point P with rotation rate $\mathbf{\Omega}$. In this sketch the rate of rotation vector $\mathbf{\Omega}$ points vertically out of the paper.

The rotation rate $\mathbf{\Omega}$ is equal to half the *vorticity* $\boldsymbol{\omega}$, i.e.

$$2\ \mathbf{\Omega} = \boldsymbol{\omega} = \text{curl}\ \ \mathbf{u} = \nabla \times \mathbf{u} = \begin{pmatrix} (\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}) \\ (\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}) \\ (\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}) \end{pmatrix} \tag{2.10}$$

- The diagonal entries of the rate of strain tensor ϵ_{ij} represent the extensional rate of strain in the direction of the three cartesian coordinate axes, as illustrated in Fig. 2.2, e.g. $Ds_1/Dt = e_{11} = \partial u_1/\partial x_1$

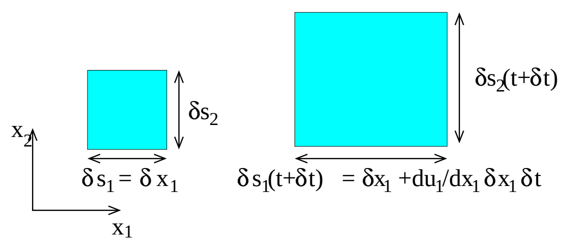

Figure 2.2: A rectangular block of fluid undergoes a purely extensional deformation which changes the lengths of the material lines parallel to the coordinate axes.

- The off-diagonal entries of the rate of strain tensor ϵ_{ij} represent the shear rate of strain (in fact, they are equal to half the shear rate in the appropriate directions; see Fig. 2.3).

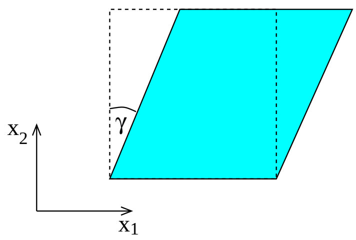

Figure 2.3: Sketch illustrating the shearing of an initially rectangular block of fluid at a rate $D\gamma/Dt = 2\ e_{12} = (\partial u_1/\partial x_2 + \partial u_2/\partial x_1)$.

2.4 The equation of continuity

- Mass conservation requires that the rate at which mass is transported over the surface ∂V of a spatially fixed volume V must be equal to the rate of change of mass in this volume. This physical statement can be formulated in an integral or a differential form:

- The integral form of the equation of continuity is given by

$$\int_V \frac{d\rho}{dt}\ dV + \oint_{\partial V} \rho u_i n_i\ dS = 0, \tag{2.11}$$

  or in symbolic form

$$\int_V \frac{d\rho}{dt}\ dV + \oint_{\partial V} \rho \mathbf{u} \cdot \mathbf{n}\ dS = 0, \tag{2.12}$$

  where ρ is the density of the fluid (i.e. the mass per unit volume), and $\mathbf{n}$ is the outer unit normal on the surface ∂V of the spatially fixed volume V (note that $\mathbf{u} \cdot \mathbf{n} < 0$ corresponds to an inflow).

- The corresponding differential form of the equation of continuity can be derived by applying the integral statement to an infinitesimally small block of fluid. The result is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0. \tag{2.13}$$

  Using the material derivative introduced in (2.6), this expression can be rewritten as

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0. \tag{2.14}$$

- The latter equation shows that for incompressible fluids (i.e. fluids for which the density of material fluid elements is constant and thus $D\rho/Dt = 0$), the equation of continuity presents a purely kinematic constraint on the velocity field, namely

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.15}$$

  or in symbolic form

$$\text{div}\ \mathbf{u} = 0 \qquad \text{or} \qquad \nabla \cdot \mathbf{u} = 0. \tag{2.16}$$

Chapter 3

Stress, Cauchy's equation and the Navier-Stokes equations

3.1 The concept of traction/stress

- Consider the volume of fluid shown in the left half of Fig. 3.1. The volume of fluid is subjected to distributed external forces (e.g. shear stresses, pressures etc.). Let $\Delta\mathcal{F}$ be the resultant force acting on a small surface element ΔS with outer unit normal $\mathbf{n}$, then the traction vector $\mathbf{t}$ is defined as:

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{\mathcal{F}}}{\Delta S} \tag{3.1}$$

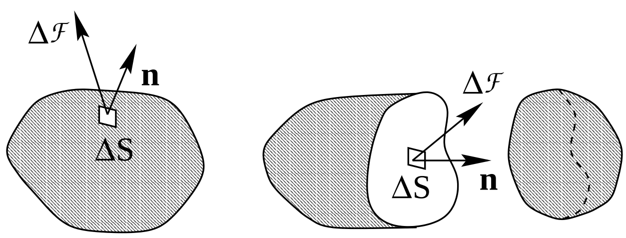

Figure 3.1: Sketch illustrating traction and stress.

- The right half of Fig. 3.1 illustrates the concept of an (internal) stress $\mathbf{t}$ which represents the traction exerted by one half of the fluid volume onto the other half across a ficticious cut (along a plane with outer unit normal $\mathbf{n}$) through the volume.

3.2 The stress tensor

- The stress vector $\mathbf{t}$ depends on the spatial position in the body and on the orientation of the plane (characterised by its outer unit normal $\mathbf{n}$) along which the volume of fluid is cut:

$$t_i = \tau_{ij} n_j, \tag{3.2}$$

where $\tau_{ij} = \tau_{ji}$ is the symmetric *stress tensor*.

- On an infinitesimal block of fluid whose faces are parallel to the axes, the component τ_{ij} of the stress tensor represents the traction component in the positive i-direction on the face $x_j = const.$ whose outer normal points in the positive j-direction (see Fig. 3.2).

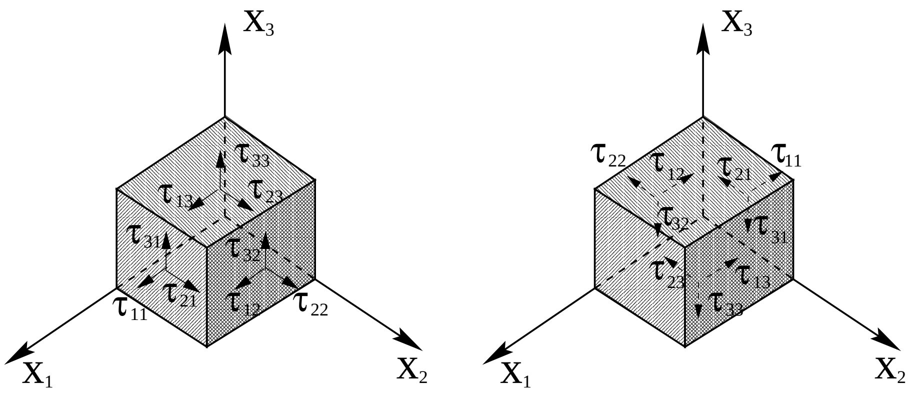

Figure 3.2: Sketch illustrating the components of the stress tensor.

3.3 Examples for simple stress states

- Hydrostatic pressure: $\tau_{ij} = -P_0\ \delta_{ij}$; note that $t_i = \tau_{ij}n_j = -P_0\ \delta_{ij}n_j = -P_0\ n_i$, i.e. the stress on any surface is normal to the surface and 'presses against it' (i.e. acts in the direction opposite to the outer normal vector) which is precisely what we expect a pure pressure to do; see left half of Fig. 3.3
- Pure shear stress: E.g. $\tau_{12} = \tau_{21} = T_0$, $\tau_{ij} = 0$ otherwise; see right half of Fig. 3.3. This sketch also illustrates that the symmetry of the stress tensor is related to the balance of moments: If τ_{21} were not equal to τ_{12} (i.e. if the tangential stress acting on the vertical faces was not equal to the tangential stress acting on the horizontal ones) then the block would rotate about the x_3 axis.

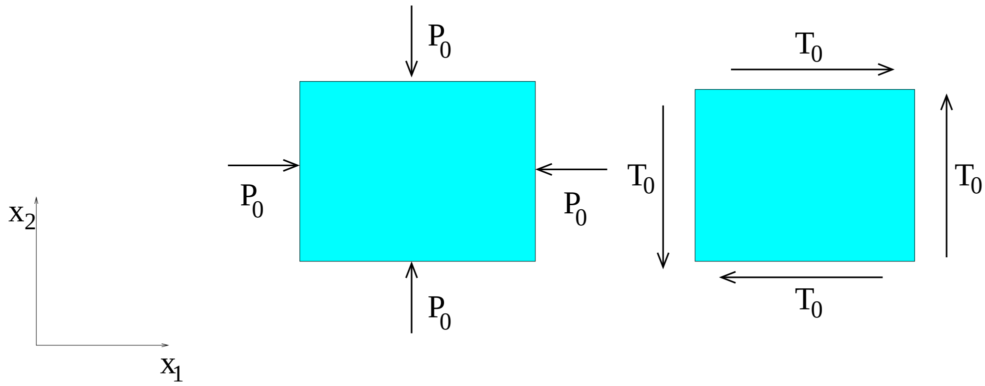

Figure 3.3: Simple stress states: Hydrostatic pressure (left) and pure shear stress (right).

3.4 Cauchy's equation

- Cauchy's equation is obtained by considering the equation of motion ('sum of all forces = mass times acceleration') of an infinitesimal volume of fluid. For a fluid which is subject to a body force (a force per unit mass) F_i, Cauchy's equation is given by

$$\rho a_i = \rho F_i + \frac{\partial \tau_{ij}}{\partial x_j}, \tag{3.3}$$

where ρ is the density of the fluid. a_i is the acceleration of the fluid, given by (2.5), therefore Cauchy's equation can also be written as

$$\rho \frac{Du_i}{Dt} = \rho F_i + \frac{\partial \tau_{ij}}{\partial x_j} \tag{3.4}$$

or

$$\rho\left(\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k}\right) = \rho F_i + \frac{\partial \tau_{ij}}{\partial x_j}. \tag{3.5}$$

- Note that Cauchy's equation is valid for *any* continuum (not just fluids!) provided its deformation is described by an Eulerian approach.

3.5 The constitutive equations for a Newtonian incompressible fluid

- In chapter 2 we derived a quantity (the rate of strain tensor ϵ_{ij}) which provides a mathematical description of the rate of deformation of the fluid.
- Cauchy's equation provides the equations of motion for the fluid, provided we know what state of stress (characterised by the stress tensor τ_{ij}) the fluid is in.
- The constitutive equations provide the missing link between the rate of deformation and the resulting stresses in the fluid.
- A large number of practically important fluids (e.g. water and oil) are incompressible and exhibit a linear relation between the shear rate of strain and the shear stresses. These fluids are known as 'Newtonian Fluids' and their constitutive equation is given by

$$\tau_{ij} = -p\delta_{ij} + 2\mu\epsilon_{ij}, \tag{3.6}$$

or, using the definition of the rate of strain tensor,

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{3.7}$$

where p is the pressure in the fluid and μ is the 'dynamic viscosity', a quantity that has to be determined experimentally.

- Note that there are also many fluids which do not behave as Newtonian fluids and have different constitutive equations (e.g. toothpaste, mayonaise). Not very imaginatively, these are often called 'Non-Newtonian Fluids' – the behaviour of these fluids is covered in a different lecture.

3.6 The Navier-Stokes equations for incompressible Newtonian fluids

- We insert the constitutive equations for an incompressible Newtonian fluid into Cauchy's equations and obtain the famous Navier-Stokes equations

$$\rho\left(\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k}\right) = \rho F_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2}, \tag{3.8}$$

or symbolically

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = \rho\mathbf{F} - \nabla p + \mu\nabla^2\mathbf{u}. \tag{3.9}$$

Dividing the momentum equations by ρ provides an alternative form

$$\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2}, \tag{3.10}$$

where $\nu = \mu/\rho$ is the 'kinematic viscosity'.

- In combination with the equation of continuity

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{3.11}$$

or symbolically

$$\nabla \cdot \mathbf{u} = 0, \tag{3.12}$$

the three momentum equations form a system of four coupled nonlinear, partial differential equations of parabolic type (second order in space and first order in time) for the three velocity components u_i and the pressure p.

The governing equations in selected coordinate systems

Rectangular cartesian coordinates

The rate of strain tensor

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}$$

where

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} \qquad \epsilon_{xy} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

$$\epsilon_{yz} = \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] \qquad \epsilon_{zx} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]$$

The vorticity

$$\omega = \operatorname{curl}\mathbf{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

The Navier Stokes equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla^2 u,$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\nabla^2 v,$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w,$$

$$\operatorname{div}\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The Laplace operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Cylindrical Polar Coordinates

Relation to Cartesian coordinates:

$$\begin{aligned} x &= r\cos\varphi, \\ y &= r\sin\varphi, \\ z &= z \end{aligned}$$

Velocity components:

$$u = u_r, \qquad v = u_\varphi, \qquad w = u_z$$

The rate of strain tensor

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{rr} & \epsilon_{r\varphi} & \epsilon_{rz} \\ \epsilon_{\varphi r} & \epsilon_{\varphi\varphi} & \epsilon_{\varphi z} \\ \epsilon_{zr} & \epsilon_{z\varphi} & \epsilon_{zz} \end{pmatrix}$$

where

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \qquad \epsilon_{\varphi\varphi} = \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{u}{r}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} \qquad \epsilon_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \varphi}\right]$$

$$\epsilon_{\varphi z} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial w}{\partial \varphi} + \frac{\partial v}{\partial z}\right] \qquad \epsilon_{rz} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right]$$

The vorticity

$$\omega = \operatorname{curl}\mathbf{u} = \left(\frac{1}{r}\frac{\partial w}{\partial \varphi} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{1}{r}\frac{\partial u}{\partial \varphi}\right).$$

The Navier Stokes equations

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \varphi} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} &= -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \varphi}\right], \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \varphi} + w\frac{\partial v}{\partial z} + \frac{uv}{r} &= -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right], \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \varphi} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w, \end{aligned}$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0.$$

The Laplace operator

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

Spherical Polar Coordinates

Relation to Cartesian coordinates:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta\cos\varphi, \\ z &= r\sin\theta\sin\varphi \end{aligned}$$

Velocity components:

$$u = u_r, \qquad v = u_\theta, \qquad w = u_\varphi$$

The rate of strain tensor

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{r\varphi} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta\varphi} \\ \epsilon_{\varphi r} & \epsilon_{\varphi\theta} & \epsilon_{\varphi\varphi} \end{pmatrix}$$

where

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \qquad \epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}$$

$$\epsilon_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi} + \frac{u}{r} + \frac{v\cot\theta}{r} \qquad \epsilon_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta}\right]$$

$$\epsilon_{\theta\varphi} = \frac{1}{2}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{w}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v}{\partial \varphi}\right] \qquad \epsilon_{\varphi r} = \frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi} + r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right]$$

The vorticity

$$\omega = \operatorname{curl}\mathbf{u} = \left(\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(w\sin\theta) - \frac{\partial v}{\partial \varphi}\right],\ \ \frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rw),\ \ \frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{1}{r}\frac{\partial u}{\partial \theta}\right).$$

The Navier Stokes equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial u}{\partial \varphi} - \frac{v^2+w^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\nabla^2 u - \frac{2u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{2v\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial w}{\partial \varphi}\right],$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial v}{\partial \varphi} + \frac{uv}{r} - \frac{w^2\cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \theta} + \nu\left[\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial w}{\partial \varphi}\right],$$

$$\begin{aligned} \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial w}{\partial \varphi} + \frac{uw}{r} - \frac{vw\cot\theta}{r} &= \\ -\frac{1}{\rho r\sin\theta}\frac{\partial P}{\partial \varphi} + \nu\left[\nabla^2 w - \frac{w}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial u}{\partial \varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v}{\partial \varphi}\right]&, \end{aligned}$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi} = 0.$$

The Laplace operator

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}.$$

Chapter 4

Boundary and initial conditions

4.1 Initial conditions

- For time-dependent problems, an initial condition for the velocity field, i.e. $u_i(x_j, t = 0)$ has to be specified.

4.2 Boundary conditions

- Fig. 4.1 shows a selection of common boundary conditions for flow problems.

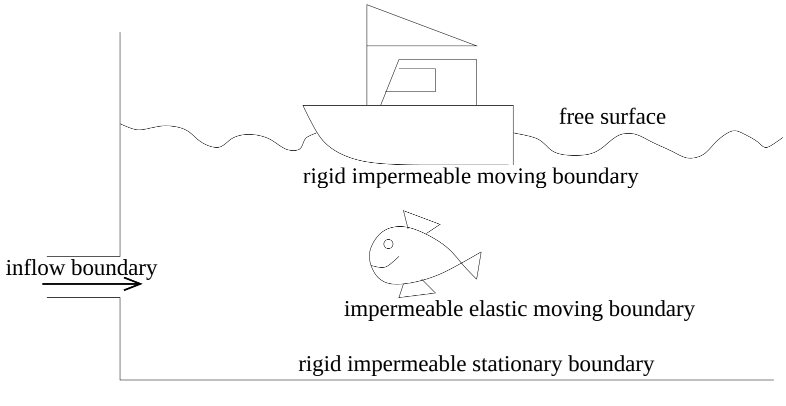

Figure 4.1: Common boundary conditions.

4.2.1 Inflow/outflow boundary conditions

- In many applications, we are only interested in the behviour of the fluid in a small region (for instance, if we want to study the ventilation in a room, it would be impractical to include the earth's entire atmosphere into the model. We would only model the room and treat its interaction with the 'rest of the world' via inflow boundary conditions – e.g. by prescribing the wind velocity through an open window). Hence at inflow (or outflow) boundaries we prescribe the velocity, i.e.

$$u_i = v_i, \tag{4.1}$$

where v_i is a prescribed function.

4.2.2 Solid surfaces

- Most solid surfaces are impermeable to fluid and the fluid 'sticks' to their surfaces. Hence, there is no slip and no penetration, and the fluid particles on the wall move with the velocity of the wall:

$$u_i = w_i, \tag{4.2}$$

where w_i is the (known) velocity of the impermeable wall.

- In the special case where the walls are stationary we have

$$u_i = 0. \tag{4.3}$$

4.2.3 Free surfaces

- Free surfaces occur at the interface between two fluids. Such interfaces require two boundary conditions to be applied: (i) A kinematic condition which relates the motion of the free interface to the fluid velocities at the free surface and (ii) a dynamic condition which is concerned with the force balance at the free surface.

(i) The kinematic boundary condition

– The position of a free surface can always be given in implicit form as $F(x_j, t) = 0$. For instance, in Fig. 4.2 the height of the free surface above the x-axis is specified as $y = h(x, t)$ and an appropriate function $F(x, y, t)$ would be given by $F(x, y, t) = h(x, t) - y$.

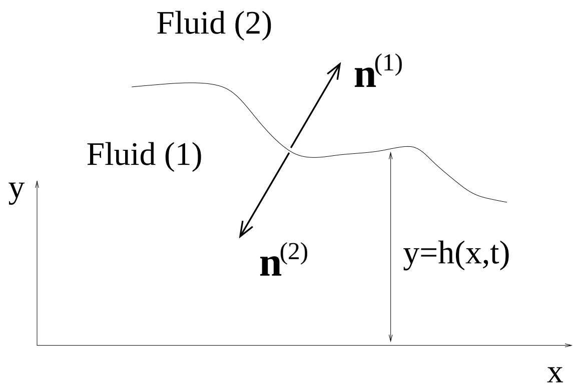

Figure 4.2: Sketch illustrating the conditions at a free surface formed by the interface between two fluids.

– Fluid particles on the free surface always remain part of the free surface, therefore we must have

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u_k \frac{\partial F}{\partial x_k}. \tag{4.4}$$

This is the kinematic boundary condition.

– For surfaces whose position is described in the form $z = h(x, y, t)$, the kinematic boundary condition becomes

$$w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}, \tag{4.5}$$

where u, v, w are the velocities in the x, y, z directions, respectively.

– For steady problems, we have $\partial F/\partial t = 0$ and the kinematic boundary condition can be written as

$$u_i n_i = 0 \quad \text{or symbolically} \quad \mathbf{u} \cdot \mathbf{n} = 0, \tag{4.6}$$

where $\mathbf{n}$ is the outer unit normal on the free surface. This condition implies that there is no flow through the free surface (but there can be a flow tangential to it!).

(ii) The dynamic boundary condition

– The dynamic boundary condition requires the stress to be continuous across the free surface which separates the two fluids (air and water in Fig. 4.1). The traction exerted by fluid (1) onto fluid (2) is equal and opposite to the traction exerted by fluid (2) on fluid (1). Therefore we must have $\mathbf{t}^{(1)} = -\mathbf{t}^{(2)}$. Since $\mathbf{n}^{(1)} = -\mathbf{n}^{(2)}$ (see Fig. 4.2) we obtain the dynamic boundary condition

$$\tau_{ij}^{(1)} n_j = \tau_{ij}^{(2)} n_j, \tag{4.7}$$

where we can use either $\mathbf{n}^{(1)}$ or $\mathbf{n}^{(2)}$ as the unit normal.

– On curved surfaces, surface tension can create a pressure jump across the free surface. The surface tension induced pressure jump is given by

$$\Delta p = \sigma \kappa. \tag{4.8}$$

In this expression σ is the surface tension of the fluid and κ is equal to twice the mean curvature of the free surface, i.e.

$$\kappa = \frac{1}{R_1} + \frac{1}{R_2}, \tag{4.9}$$

where R_1 and R_2 are the principal radii of curvature of the surface (for instance, $\kappa = 2/a$ for a spherical drop of radius a and $\kappa = 1/a$ for a circular jet of radius a). Surface tension acts like a tensioned membrane at the free surface and tries to minimise the surface area. Hence the pressure inside a spherical drop (or inside a circular liquid jet) tends to be higher than the pressure in the surrounding medium.

– If surface tension is important, the dynamic boundary condition has to be modified to

$$\tau_{ij}^{(1)} n_j^{(1)} + \sigma \kappa n_i^{(1)} = \tau_{ij}^{(2)} n_j^{(1)}, \tag{4.10}$$

where $\kappa > 0$ if the centres of curvature lie inside fluid (1).

4.2.4 Other boundary conditions

- Other boundary conditions can occur in special applications. For instance, the presence of an elastic boundary leads to fluid-structure interaction problems in which the fluid velocity has to be equal to the velocity of the elastic wall, while the elastic wall deforms in response to the traction that the fluid exerts on it. At porous walls, the no-penetration condition no longer holds: the volume flux into the wall is often proportional to the pressure gradient at the porous surface. Non-uniformly distributed surfactants (substances which reduce the surface tension) can induce tangential stresses at free surfaces, etc.

4.3 Further remarks

- For an incompressible fluid, the boundary conditions need to fulfill the overall consistency condition

$$\oint_{\partial V} u_i n_i \, dS = 0, \tag{4.11}$$

where ∂V is the surface of the spatially fixed volume in which the equations are solved.

- If there are no free surfaces (and associated dynamic boundary conditions), the pressure is only defined up to an arbitary constant as only the pressure gradient (but not the pressure itself) appears in the Navier-Stokes equations.

- For initial value problems, the initial velocity field (at $t = 0$) already has to fulfill the incompressibility constraint.

These remarks are particularly important for the numerical solution of the Navier-Stokes equations.

Chapter 5

Parallel Flows

5.1 The parallel flow equations

- The main difficulty in the solution of the Navier Stokes equations arises from their nonlinearity. There are, however, situations in which the nonlinear terms vanish identically.
- This happens (for instance) if the flow is *unidirectional*. If this is the case then we can choose our coordinate system such that the x-axis is aligned with the flow and the velocity field has the form $\mathbf{u} = u(x, y, z, t)\ \mathbf{e}_x$.
- Inserting this assumption into the Navier Stokes and continuity equation shows that this is only possible if
$$\mathbf{u} = u(y, z, t)\ \mathbf{e}_x, \tag{5.1}$$
i.e. if the velocity is independent of the streamwise coordinate.
- The flow governed by the following three linear equations
$$\rho \frac{\partial u}{\partial t} = \rho F_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \tag{5.2}$$
$$0 = \rho F_y - \frac{\partial p}{\partial y} \tag{5.3}$$
and
$$0 = \rho F_z - \frac{\partial p}{\partial z}. \tag{5.4}$$

5.2 The parallel flow equations without body force

- If the body force vanishes (i.e. $F_x = F_y = F_z = 0$) it can be shown that $p = p(x, t)$ and the pressure gradient has to have the form
$$\nabla p = G\ \mathbf{e}_x \tag{5.5}$$
where G is a constant. (If the pressure gradient has any other form, then no parallel flow is possible).
- In this case, the only non-trivial equation is the x-momentum equation which becomes
$$\rho \frac{\partial u}{\partial t} = -G + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{5.6}$$

Chapter 6

Curvilinear Coordinates

- For flows in circular (or spherical) geometries, cartesian coordinates are not the most convenient coordinate system to work in.

- The transformation of the Navier-Stokes and continuity equations to other coordinate systems is straightforward (if messy) and is based on a simple coordinate transformation, such as $x = r\cos\varphi,\ y = r\sin\varphi$ for the transformation between 2D cartesian and plane cylindrical polar coordinates. Following the usual rules, we can transform differential operators to the new coordinates, e.g.
$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}. \tag{6.1}$$

- We also need to transform vectors to the new coordinate system by decomposing them into the new basis vectors, e.g.,
$$\mathbf{u} = u_x\mathbf{e}_x + u_y\mathbf{e}_y = u_r\mathbf{e}_r + u_\varphi\mathbf{e}_\varphi, \tag{6.2}$$
where u_r and u_φ are the velocity components in radial and circumferential direction.

- Note that in curvilinear coordinates, the basis vectors depend on the coordinates (e.g. $\mathbf{e}_r = (\cos\varphi, \sin\varphi)$). Hence, any differential operator acting on a vector acts on the basis vectors as well as the components themselves. The resulting vector then has to be decomposed into the basis vectors. This makes the resulting expressions considerably more complicated than their equivalents in cartesian coordinates (see the Navier Stokes equations in curvilinear coordinates in chapter 3).

- Provided we restrict ourselves to orthogonal coordinate systems (such as cylindrical and spherical polar coordinates) we can still use the index notation and the summation convention. For instance, in plane cylindrical polars the traction boundary condition can be written as
$$t_i = (-p\delta_{ij} + 2\mu e_{ij})n_j \quad \text{where } i,j \text{ represent the coordinate directions } r \text{ and } \varphi. \tag{6.3}$$

- Thus, for instance, the r-component of the traction $\mathbf{t} = t_r\mathbf{e}_r + t_\varphi\mathbf{e}_\varphi$ is given by
$$t_r = -pn_r + 2\mu(e_{rr}n_r + e_{r\varphi}n_\varphi) \tag{6.4}$$
where $\mathbf{n} = n_y\mathbf{e}_r + n_\varphi\mathbf{e}_\varphi$ is the outer unit normal on the fluid, decomposed into the cylindrical basis vectors and the e_{ij} are the components of the rate of strain tensor in plane cylindrical polars as given in chapter 3.

Chapter 7

Dimensional Analysis and Scaling

7.1 Dimensional Analysis and Scaling

- See the copies of the OHP transparencies.

7.2 Similarity Solutions

- Many phenomena in fluid mechanics exhibit self-similar behaviour. A simple example for this is given by a velocity profile $u(y,t)$ which has the same 'shape' for all values of time t. In that case, the solution must have the special form

$$u(y,t) = a(t)\ f\left(\frac{y}{b(t)}\right), \tag{7.1}$$

  where the function $a(t)$ changes the 'amplitude' of $u(y,t)$ while $b(t)$ provides a time-dependent scaling for the y-coordinate, making the velocity profile 'wider' (for $b(t) > 1$) or 'narrower' (for $b(t) < 1$).

- The existence of similarity variables is also familiar from traveling wave problems in which the solution has the form $u(y,t) = f(y - Ut)$. This solution represents a wave of shape $f(y)$ traveling in the positive y-direction with velocity U. The traveling wave coordinate $\eta = y - Ut$ plays the role of a similarity variable.

- In general, similarity solutions are characterised by the requirement that at least one independent variable only occurs in a certain combination with other independent variables. This often simplifies the mathematical analysis, for instance, by transforming PDEs into ODEs.

- The search for suitable similarity variables is often aided by dimensionality considerations.

- The choice of the similarity variable is usually not unique. For instance, a function $f(\eta)$ could also be regarded as a function $F(\eta^2)$ – obviously, both η and η^2 are perfectly acceptable choices. Typically, one tries to keep the similarity variable linear in the spatial coordinate, as in (7.1) where $\eta = y/b(t)$.

- Similarity solutions only 'work' if the boundary and initial conditions can also be formulated in terms of the similarity variable. If this is not the case, the similarity 'solution' might (!) still represent a useful approximation to the exact solution.

Chapter 8

The Streamfunction and Vorticity

- For 2D incompressible flows, it is possible to recast the Navier-Stokes equations in an alternative form in terms of the streamfunction and the vorticity.
- In many applications, the streamfunction-vorticity form of the Navier Stokes equations provides better insight into the physical mechanisms driving the flow than the 'primitive variable' formulation in terms of u, v and p.
- The streamfunction and vorticity formulation is also useful for numerical work since it avoids some problems resulting from the discretisation of the continuity equation.

Unless specifically stated, all results in this chapter are restricted to 2D incompressible flows.

8.1 The Streamfunction

- The streamfunction is defined as

$$\psi_A(P) = \int_A^P \mathbf{u} \cdot \mathbf{n} \, ds, \tag{8.1}$$

where the integral has to be evaluated along a line from the arbitrary but fixed point A to point P. $\mathbf{n}$ is the unit normal on the line from A to P. We regard $\psi_A(P)$ as a function of the location of point P.

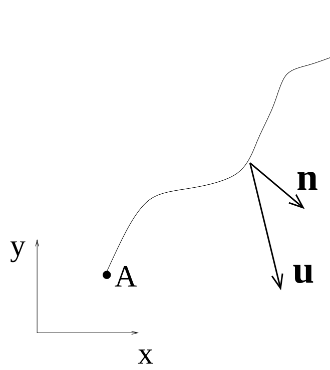

Figure 8.1: Sketch illustrating the definition of the streamfunction.

- The sketch in Fig. 8.1 shows that $\mathbf{u} \cdot \mathbf{n}$ is equal to the component of the velocity $\mathbf{u}$ that crosses the line AP. Therefore $\psi_A(P)$ represents the volume flux (per unit depth in the z-direction) through the line between A and P.

- Evaluating $\psi_A(P)$ along two different paths and invoking the integral form of the incompressibility constraint shows that $\psi_A(P)$ is path-independent, i.e. its value only depends on the locations of the points A and P.
- Changing the position of point A only changes $\psi_A(P)$ by a constant. It turns out that for all applications such changes are irrelevant. It is therefore common to suppress the explicit reference to A. Hence, we regard $\psi_A(P)$ as a function of the spatial coordinates only, i.e. $\psi_A(P) = \psi(P) = \psi(x, y)$.
- Streamlines are lines which are everywhere tangential to the velocity field, i.e. $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal to the streamline. Hence the streamfunction ψ is constant along streamlines.
- Note that stationary impermeable boundaries are also characterised by $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal on the boundary. Therefore, ψ is also constant along such boundaries.
- Invoking the integral incompressibility constraint for an infinitesimally small triangle shows that ψ is related to the two cartesian velocity components u and v via

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{8.2}$$

- Similarly, in plane cylindrical polars, the velocity components are given by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad \text{and} \quad u_\varphi = -\frac{\partial \psi}{\partial r}. \tag{8.3}$$

- Flows which are specified by a streamfunction automatically satisfy the continuity equation since

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0. \tag{8.4}$$

- For 2D flows, the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ only has one non-zero component (in the z-direction), i.e. $\boldsymbol{\omega} = \omega \mathbf{e}_z$ where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{8.5}$$

  Using the definition of the velocities in terms of the streamfunction shows that

$$\omega = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) \tag{8.6}$$

  and therefore

$$\omega = -\nabla^2 \psi, \tag{8.7}$$

  where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the 2D Laplace operator.

8.2 The Streamfunction-Vorticity form of the Navier-Stokes equations

- Straightforward algebraic manipulation of the 3D momentum equations transforms them into the *vorticity transport equation*

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega} \tag{8.8}$$

  (see the separate handout for the derivation; this equation is valid in 3D).

- This equation shows that the rate of change of the vorticity of material particles, $D\boldsymbol{\omega}/Dt$, is controlled by 'vortex stretching' (described by $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$; this is a familiar result from inviscid fluid mechanics) and by diffusion (described by $\nu \nabla^2 \boldsymbol{\omega}$). The diffusion of vorticity only occurs in viscous flows.

- For 2D flows, vortex stretching is absent since $\mathbf{u} = u(x,y)\ \mathbf{e}_x + v(x,y)\ \mathbf{e}_y$ and $\boldsymbol{\omega} = \omega(x,y)\ \mathbf{e}_z$ and therefore $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u} = 0$.

- For 2D flows, the scalar vorticity transport equation

$$\frac{D\omega}{Dt} = \nu\nabla^2\omega \qquad (8.9)$$

together with the equation for the vorticity in terms of the streamfunction

$$\omega = -\nabla^2\psi \qquad (8.10)$$

and

$$u = \partial\psi/\partial y \quad \text{and} \quad v = -\partial\psi/\partial x \qquad (8.11)$$

provide the streamfunction-vorticity formulation of the Navier-Stokes equations. It consists of only two PDEs for the scalars ω and ψ rather than the three PDEs for u, v and p in the 'primitive variable' form.

- In the limit of zero Reynolds number, only one fourth-order PDE for the streamfunction ψ needs to be solved, namely the biharmonic equation

$$\nabla^4\psi = 0, \qquad (8.12)$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}. \qquad (8.13)$$

This can be shown by, e.g., taking the curl of the Stokes equations.