$$\boxed{\begin{aligned} &\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i + \rho f_i \\ &\frac{\partial u_k}{\partial x_k} = 0 \end{aligned}}$$

+ BC & IC.

N.St. are very complicated primarily because of the nonlin. terms $u_j \frac{\partial u_i}{\partial x_j}$.

These terms vanish in a number of practically relevant situations. Example:

§mt2 Parallel flows

Assume (here) that the flow is unidirectional & we align the x-axis with the direction of the flow

$$(x_1, x_2, x_3) \Longleftrightarrow (x, y, z)$$

$$(u_1, u_2, u_3) \Longleftrightarrow (u, v, w)$$

$u \neq 0 \; ; \quad v = w = 0$

Note: u can still vary as a fct. of time & space.
$u = u(x, y, z, t)$

Consequences:

Continuity:

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}} = 0$$

$$\Rightarrow u = u(y, z, t)$$

Use this in the x-mom. eqn

$$\rho\left(\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}}\right) =$$

$$\rho f_x - \frac{\partial p}{\partial x} + \mu \nabla^2 u \qquad \cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\boxed{\rho \frac{\partial u}{\partial t} = \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)} \quad (1)$$

y-comp:

$$\rho\left(\cancel{\frac{\partial v}{\partial t}}+\cancel{u\frac{\partial v}{\partial x}}+\cancel{v\frac{\partial v}{\partial y}}+\cancel{w\frac{\partial v}{\partial z}}\right)=$$

$$+\rho f_y-\frac{\partial p}{\partial y}+\cancel{\mu\nabla^2 v}$$

$$\boxed{\rho f_y=\frac{\partial p}{\partial y}} \quad (2)$$

z-comp:

$$\boxed{\rho f_z=\frac{\partial p}{\partial z}} \quad (3)$$

Note: (2) & (3) completely determine the y & z dependence of the pressure.

Note: (1)-(3) are linear!!

Special case: No body force

$\underline{f} = \underline{0}$

In that case (2) & (3) imply $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$

$p(x,t)$ (at most)

Stick this into (1) fure $u(y,z,t)$

$$\underbrace{\rho \frac{\partial u}{\partial t}}_{\substack{\text{fct of} \\ (y,z,t)}} = \underbrace{-\frac{\partial p}{\partial x}}_{\substack{\text{fct of} \\ (x,t)}} + \mu \underbrace{\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\substack{\text{fct of} \\ (y,z,t)}}$$

This requires that

$\frac{\partial p}{\partial x} \neq$ fct of x

$$\frac{\partial p}{\partial x} = G(t)$$

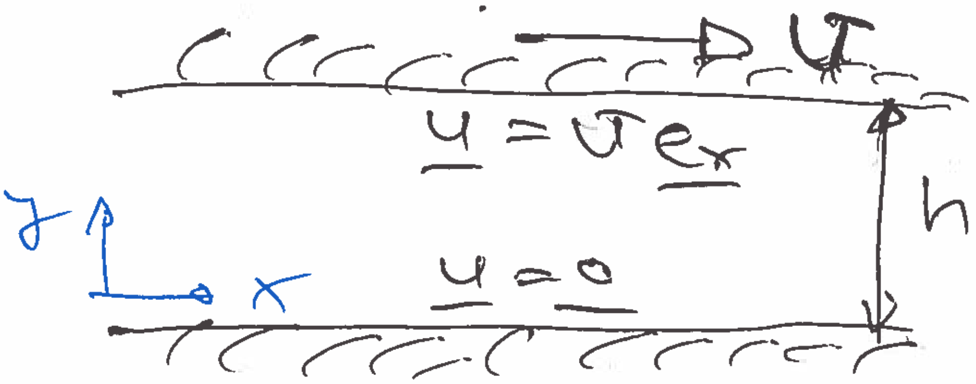
Parallel flow eqns without body force

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{G}{\rho} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\ \frac{\partial p}{\partial x} &= G(t) \\ v = w &= 0 \end{aligned}}$$

$\nu = \frac{\mu}{\rho}$ = Kinematic viscosity.

Example: Couette flow

Flow between parallel infinite plates. Upper plane moves to right with veloc U.



Assume:

- $u(y,z,t) = u(y,t)$
- Steady: $u(y,t) = u(y)$
- $\frac{\partial p}{\partial x} = G = 0$

$$\cancel{\frac{\partial u}{\partial t}} = -\cancel{\frac{G}{\rho}} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\frac{d^2 u}{dy^2} = 0$$

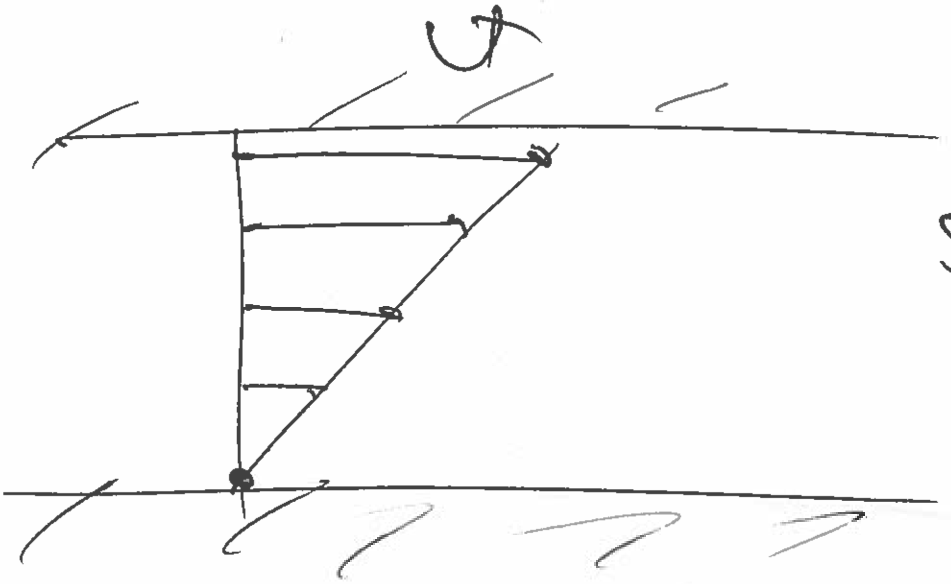
$$u(y) = A + By$$

2 BC:

$$u(y=0) = 0$$

$$u(y=h) = U$$

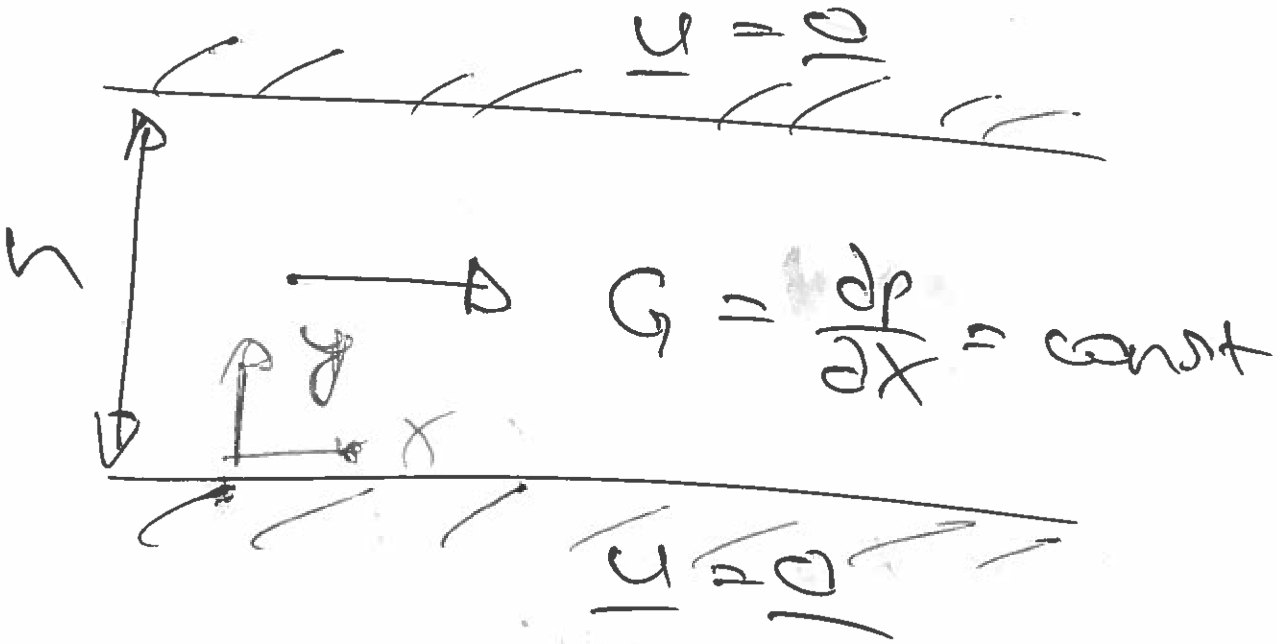
$$u(y) = U\frac{y}{h}.$$



Shear flow

Example: Poiseuille flow

pressure-driven flow in channel



Assume: $u(y,z,t) = u(y)$ as before

$$\cancel{\rho\frac{\partial u}{\partial t}} = -G + \mu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$G = \mu\frac{d^2 u}{dy^2}$$

$$u(y) = \frac{1}{2}\frac{G}{\mu}y^2 + Ay + B$$

2BCs: $u(y=0) = 0$

$u(y=h) = 0$

$$u(y) = \frac{G}{2\mu}\left(y^2 - hy\right).$$