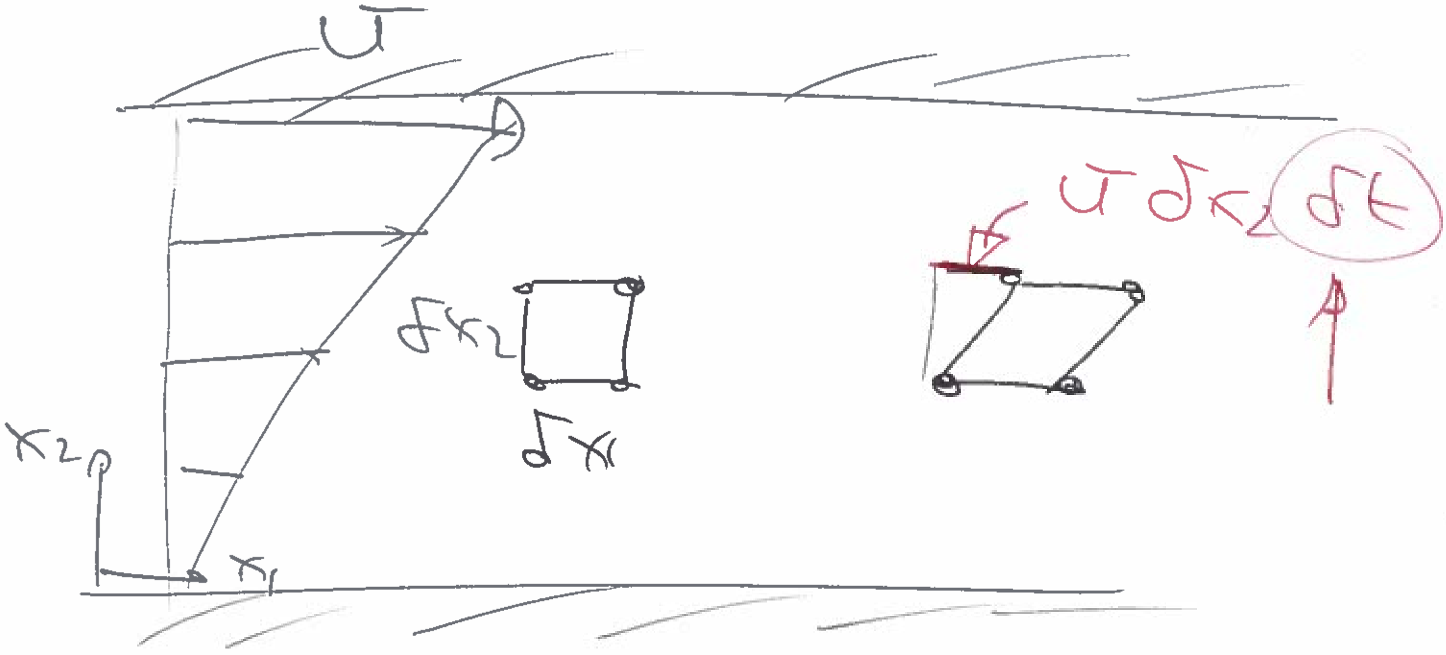
Correction to Ex. class:



$$\boxed{\begin{gathered}\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho f_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} \\ \frac{\partial u_k}{\partial x_k} = 0\end{gathered}}$$

<u>IC</u> $u_i(x_k, t=0)$ given

<u>BC:</u> Inflow / outflow / no-slip

$u_i\big|_{\partial\Omega}$ given

(iii) free surface

---

need 2 conditions

- kinematic boundary cond. (BC)
- traction BC

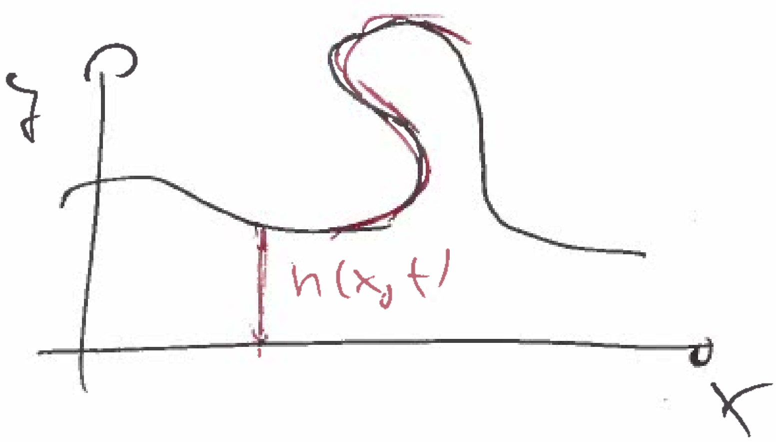
(e) Kinematic BC

The position of the free surface can always be described implicitly as

$$F(x, y, z, t) = 0 \quad \text{on the free surface}$$

At least locally this can be inverted to

$$z = h(x, y, t)$$



2D: $F(x, z, t) = 0$

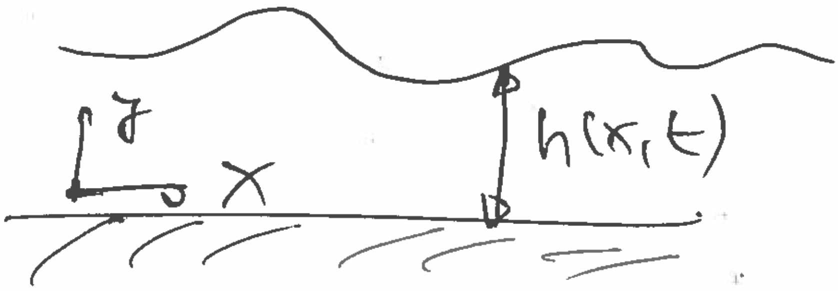
$z = h(x, t)$

Physical observation:

Fluid particles on the free surface stay on that surface. $\Rightarrow F \equiv 0$ for a particle on the surface

$$\boxed{\frac{DF}{Dt} = 0}$$

<u>Example:</u>



$$F(x,y,t) = h(x,t) - y = 0$$

for particles on surface

In that case $\frac{DF}{Dt}$:

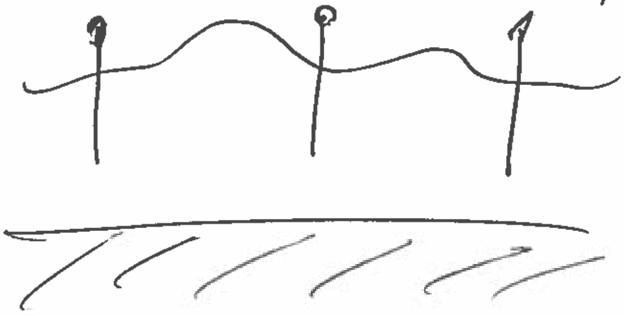
$$\left.\begin{aligned} \frac{\partial F}{\partial t} &= \frac{\partial h}{\partial t} \\ \frac{\partial F}{\partial x} &= \frac{\partial h}{\partial x} \\ \frac{\partial F}{\partial y} &= -1 \end{aligned}\right\} \frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0$$

$$\boxed{\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} - v = 0}$$

on the surface

Special case 1:

$u = 0$: only vertical velocity



$$\frac{\partial h}{\partial t} = v \quad \text{on } y = h \quad \checkmark$$

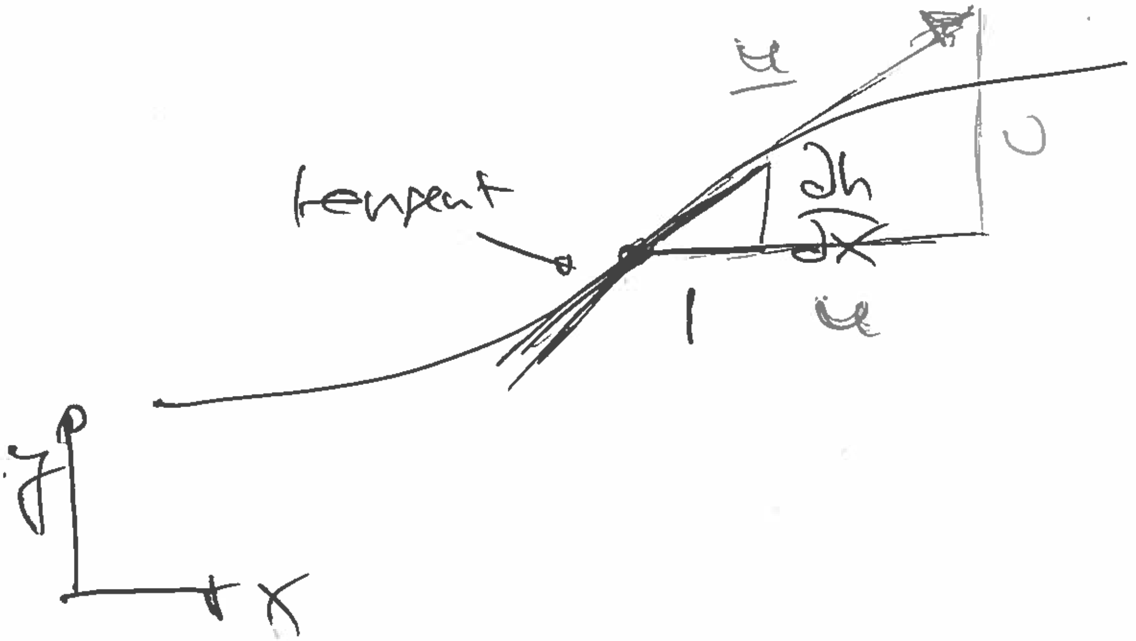
Special case 2:

Fixed free surface height

$$\frac{\partial h}{\partial t} = 0 :$$

$$u\frac{\partial h}{\partial x} = v$$

$$\frac{\partial h}{\partial x} = \frac{v}{u}$$

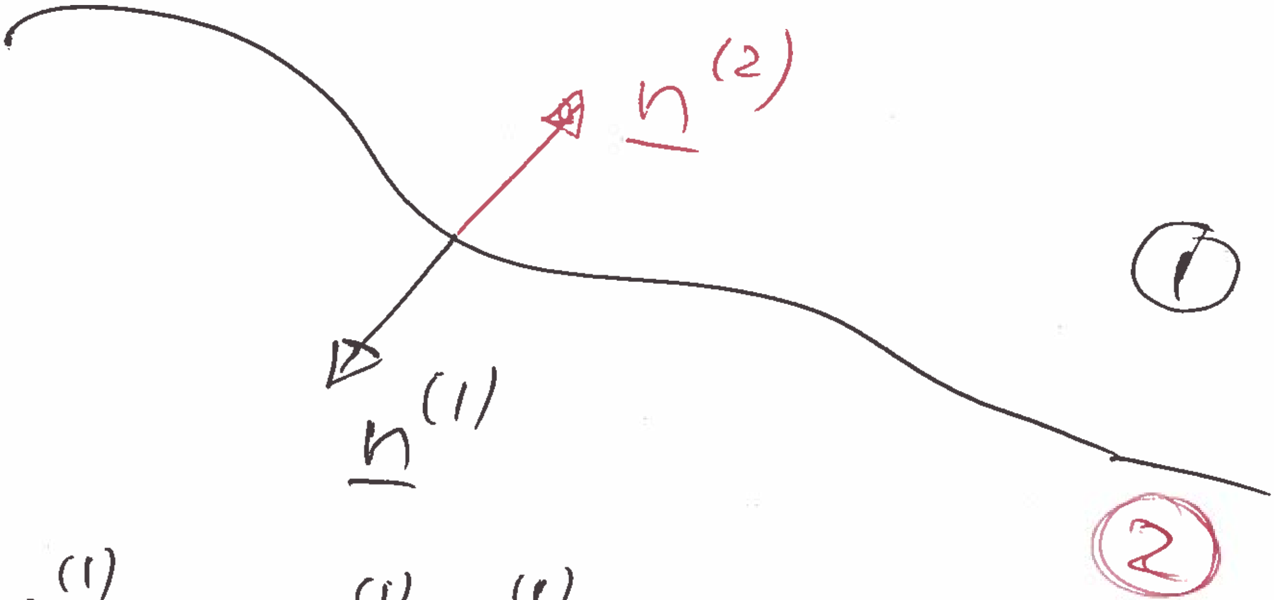


$\Rightarrow$ veloc is tangential to the surface!

(b) traction BC

- On a free surface the applied traction is equal to the stress in the fluid.
- If the free surface is the interface between two fluids the stress is continuous across the surface

6



$$t_i^{(1)} = \tau_{ij}^{(1)} n_j^{(1)}$$

$$t_i^{(2)} = \tau_{ij}^{(2)} n_j^{(2)}$$

traction applied onto fluid 2

outer unit normal on fluid 2

Action = reaction

$$\underline{t}^{(1)} = -\underline{t}^{(2)}$$

$$\underline{n}_1^{(1)} = -\underline{n}^{(2)}$$

$$\boxed{\tau_{ij}^{(1)} n_j = \tau_{ij}^{(2)} n_j}$$

where $\underline{n}$ is one of $\underline{n}^{(1)}$ & $\underline{n}^{(2)}$

<u>Example:</u> Hydrostatics

$$\tau_{ij} = -p\,\delta_{ij}$$

$$-p^{(1)} n_i = -p^{(2)} n_i$$

$$\left(p^{(1)} - p^{(2)}\right) \underline{n} = \underline{0}$$

$$p^{(1)} = p^{(2)}$$

pressure const. across the surface.