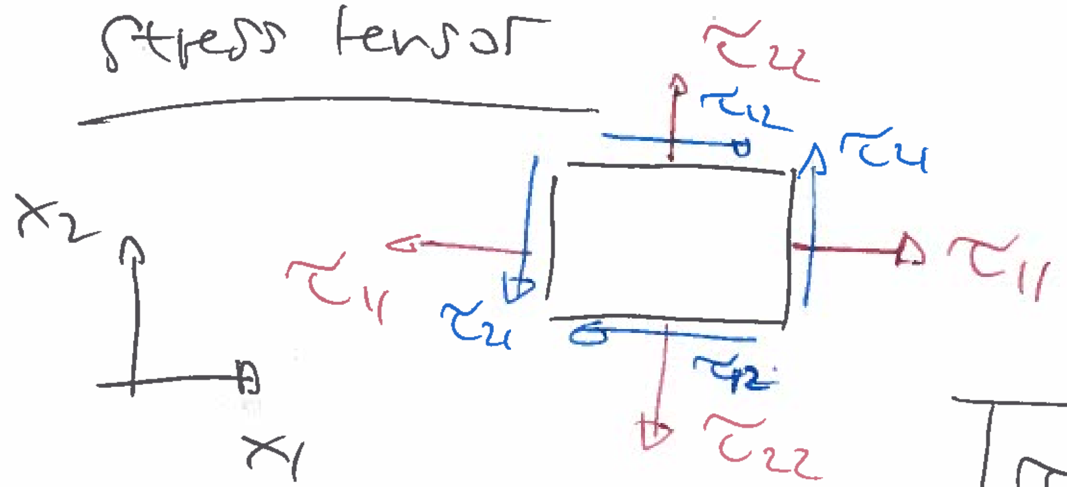
Stress tensor



$$\boxed{\tau_{ij} = \tau_{ji}}$$

$\tau_{ij}(x_k, t)$

Cauchy's eqn:

$$\frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i = \rho \frac{Du_i}{Dt} = \rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right)$$

f_i → body force (per unit mass, e.g. gravitational accel.)

Finally: Constitutive eqns

Here: Restrict ourselves to incompressible fluids

Const. eqns. provide a link between the kinematics of flow & the stress tensor.
(Requires experiments)

Observations:

Fluids:

- can generate hydrostatic pressures.
- have a resistance to shear (knife in honey!)
  $\Rightarrow$ viscosity
- do not generate internal stresses when subjected to rigid body motions

$\Rightarrow$ The stress tensor τ_{ij} should contain a hydrostatic pressure contribution & depend on the rate of strain tensor.

A wide range of incompressible fluids behave according to

$$\tau_{ij} = \underbrace{-p\,\delta_{ij}}_{\text{hydrostatic contribution}} + \underbrace{2\mu\,\varepsilon_{ij}}_{\text{viscous stresses depend linearly on } \varepsilon_{ij}}$$

μ = "dynamic viscosity" has to measured experimentally

(Newtonian fluids)

$$\tau_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

into Cauchy's eqn

$$\rho \frac{Du_i}{Dt} = \rho f_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

$$= \rho f_i + \frac{\partial}{\partial x_j}\left(-p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)$$

$$= \rho F_i - \frac{\partial p}{\partial x_i} + \mu \underbrace{\frac{\partial^2 u_i}{\partial x_j^2}}_{\nabla^2 u_i} + \mu \frac{\partial}{\partial x_i}\underbrace{\left(\frac{\partial u_j}{\partial x_j}\right)}_{\underbrace{\nabla \cdot \underline{u} = 0}_{\text{(incompr.)}}}$$

$$\boxed{\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho f_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}}$$

momentum eqns.

$$\boxed{\frac{\partial u_j}{\partial x_j} = 0}$$

continuity eqn

= Navier Stokes eqns!

A system of four nonlinear, coupled PDEs of 2^{nd} order in space for 3 veloc & 1 pressure.

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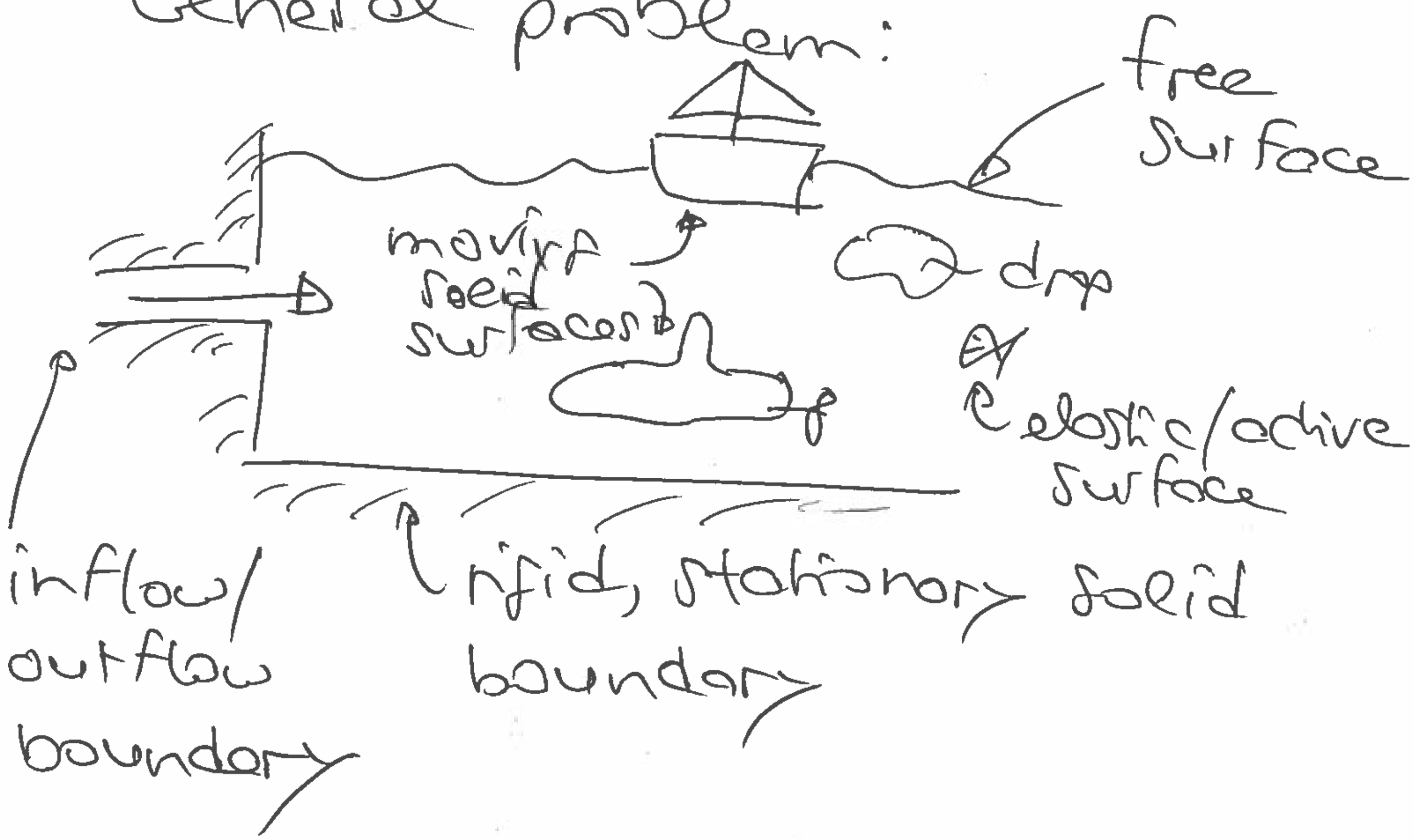
§... Boundary and initial conditions

General problem:



Initial conditions:

Need to specify the initial velocity

$u_i(x_j, t=0)$ given.

<u>Note:</u> No ICs for pressure.

Boundary conditions:

(i) Inflow/outflow BCs

$u_i = v_i$ prescribed on those boundaries.

(ii) on solid surfaces

"No slip & no penetration"

$\Rightarrow$ Solid velocity (given) = fluid velocity.

$u_i = v_i \leftarrow$ given solid velocity

Special case: stationary boundary:

$$u_i = 0.$$