Continuity eqn:

Integral:

$$\int \frac{\partial \rho}{\partial t}\, dV + \oint \rho \underline{u} \cdot \underline{n}\, dA = 0$$

Differential:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{u}) = 0$$

$\rho = \rho(x_k, t)$ density $\left[\frac{kg}{m^3}\right]$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\underbrace{(\rho u_i)} = 0$$

$$\underbrace{\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

$\frac{D\rho}{Dt}$ = rate of change of density of a fluid particle

For an incompressible fluid

$$\frac{D\rho}{Dt} = 0$$

for such fluids:

$$\frac{\partial u_i}{\partial x_i} = \operatorname{div} \underline{u} = 0$$

This is a purely kinematic constraint!

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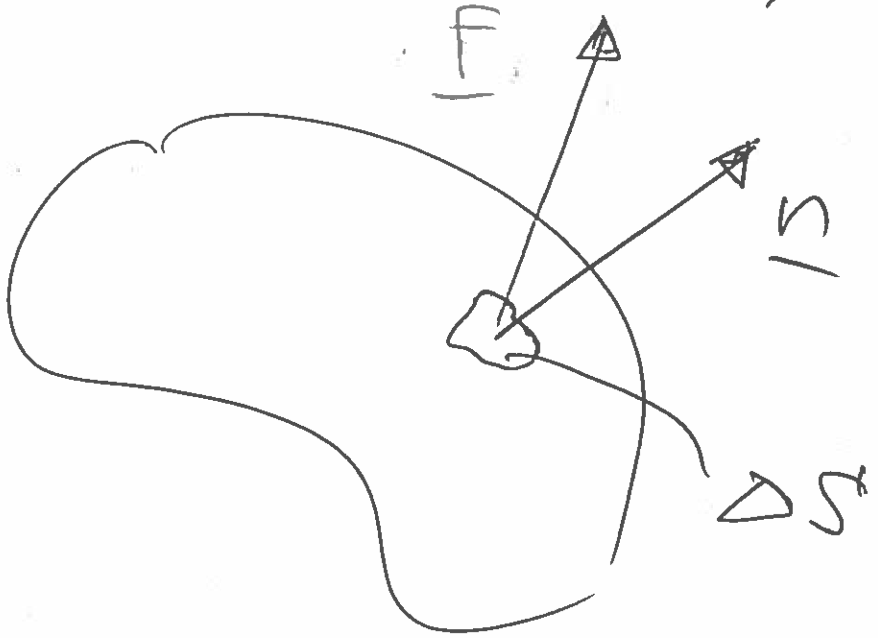
§n+1 Stress, Cauchy's eqn. the Navier-Stokes eqns

Idea: use Newton's 2nd law for the fluid to determine an eqn for the evolution of $\underline{u}$.

① The concept of stress/traction

Consider a finite blob of fluid loaded by some distributed force (pressure, shear stress, etc.).



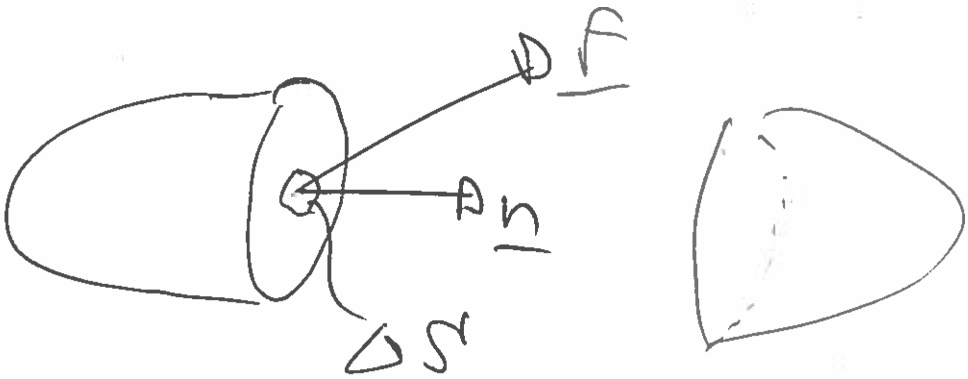
Every patch ΔS on the surface (outer unit normal $\underline{n}$) is subject to a resultant force $\underline{F}$

<u>Def:</u> Traction

$$\underline{t} = \lim_{\Delta S \to 0} \frac{\underline{F}}{\Delta S}$$ vector!

$\underline{t}$ = force per unit area exerted onto the fluid.

Similarly: Cut the blob along a plane with outer unit normal $\underline{n}$:



Here $\underline{F}$ represents the force exerted onto ΔS by the other half of the blob.

Def: Stress

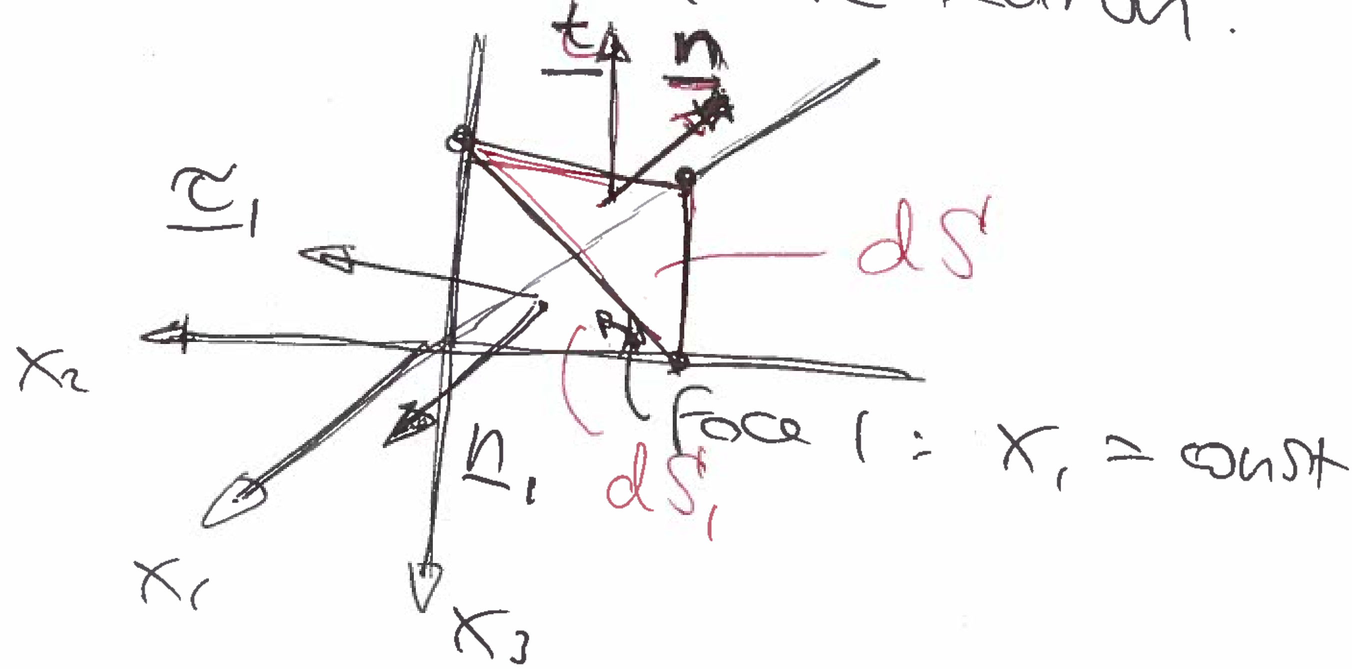
$$\underline{t} = \lim_{\Delta S \to 0} \frac{\underline{F}}{\Delta S} \quad \text{vector!}$$

Note: Stress depends on

- the position in fluid
- the direction of the "cut", i.e. on $\underline{n}$.

② The stress tensor

To examine dependence of $\underline{t}$ on $\underline{n}$ consider an infinitesimal tetrahedron:



$\underline{n}_1 = \underline{e}_1$

Face i is characterised by (e

x_i = const. & $\underline{n_i} = \underline{e_i}$.

__Trick:__ Represent the faces (have orientation & an area!) by area vectors ‖ to outer unit normals & magnitude = area.

__Then:__

$$\underline{n_i}\, dS_i + \underline{n}\, dS = \underline{0}$$

(EXERCISE)

multiply by $\underline{n_j} = \underline{e_j}$

$$\underbrace{\underline{n_i} \cdot \underline{n_j}}_{\delta_{ij}}\, dS_i + \underbrace{\underline{n} \cdot \underline{e_j}}_{n_j}\, dS = 0$$

↳ j-th comp. of $\underline{n}$

$$\boxed{dS_j = -n_j\, dS}$$

<u>Now</u>: balance of forces

$\Rightarrow$ sum of all <u>forces</u> has to be zero.

$$\underline{t}\, dS = -\underline{\tau}_j\, dS_j$$

$$\underline{t}\, \cancel{dS} = \underline{\tau}_j\, n_j\, \cancel{dS}$$

$$\boxed{t_i = \tau_{ij} n_j}$$

τ_{ij} = stress tensor.