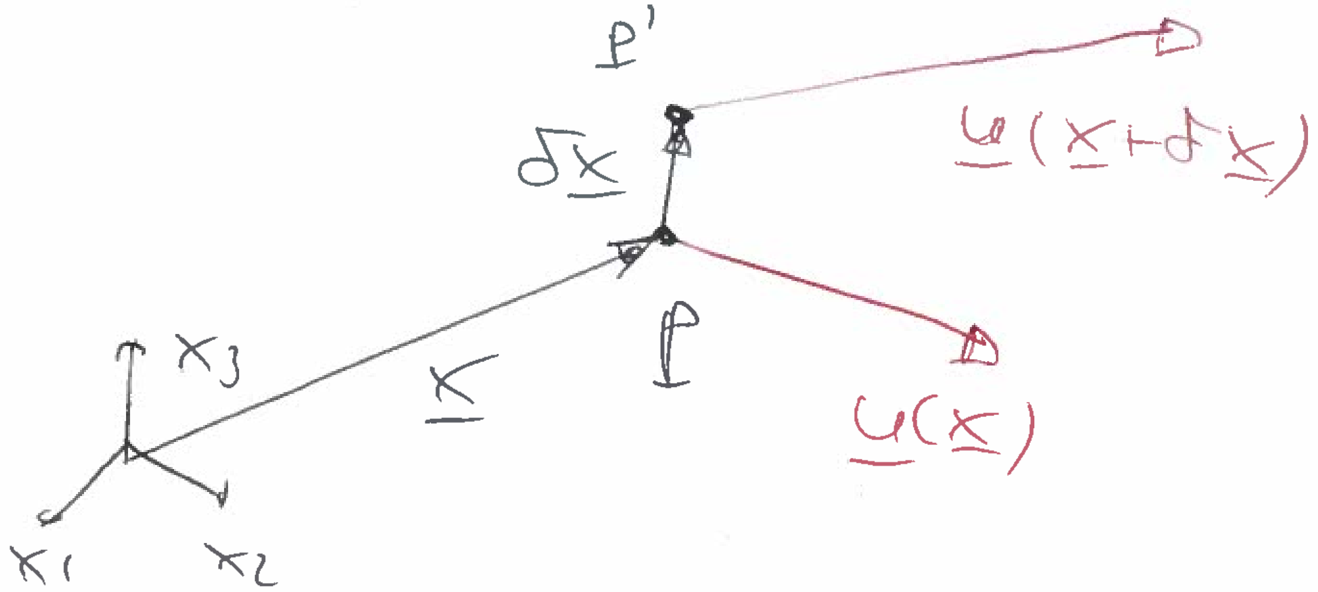
Velocity field "contains"

- translation
- rotation
- shearing
- dilation



$$\delta \underline{u} = \underline{u}(\underline{x}+\delta\underline{x}) - \underline{u}(\underline{x})$$

$\delta x_i \to 0$: $\delta x_i \to dx_i$

$$du_i = \underbrace{\frac{\partial u_i}{\partial x_j}}_{\text{"velocity gradient tensor"}} dx_j$$

"velocity gradient tensor" (3x3 matrix)

$\Rightarrow$ Translation (i.e. $\delta\underline{u} = \underline{0}$)

$$\frac{\partial u_i}{\partial x_j} = 0$$

(all particles move with the same velocity!)

So $\frac{\partial u_i}{\partial x_j}$ contains all other "modes".

To see this, split $\frac{\partial u_i}{\partial x_j}$ into sym. & anti-sym. part:

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\varepsilon_{ij} = \varepsilon_{ji}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\omega_{ij} = -\omega_{ji}}$$

$$\delta u_i = \underbrace{\varepsilon_{ij}}_{\substack{\text{rate of} \\ \text{strain tensor;} \\ \text{describes} \\ \text{shearing \&} \\ \text{dilation}}} \delta x_j + \underbrace{\omega_{ij}}_{\substack{\text{rate of} \\ \text{rotation} \\ \text{tensor; describe} \\ \text{rotation.}}} \delta x_j$$

① Rigid body rotation / vorticity

Consider the incremental change in velocity caused by

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = -\omega_{ji}$$

(antisymmetric!)

$$\delta u_i = \omega_{ij}\,\delta x_j$$

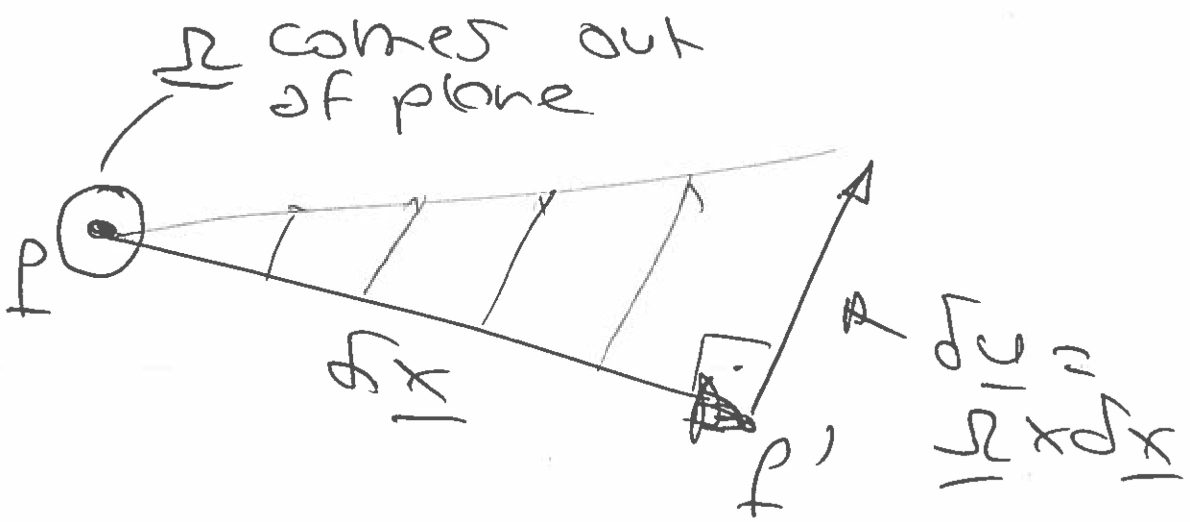
$$\begin{pmatrix}\delta u_1\\ \delta u_2\\ \delta u_3\end{pmatrix} = \begin{pmatrix}0 & -\Omega_3 & \Omega_2\\ \Omega_3 & 0 & -\Omega_1\\ -\Omega_2 & \Omega_1 & 0\end{pmatrix}\begin{pmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\end{pmatrix}$$

Can write this as

$$\delta \underline{u} = \underline{\Omega} \times \delta \underline{x}$$ where

$$\Omega = \begin{pmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{pmatrix}$$ is the rate of rotation vector.

Geom. interpretation:



$\Rightarrow$ $\underline{\Omega}$ induces rotation!

$$\underline{\Omega} = \frac{1}{2} \underbrace{\nabla \times \underline{u}}_{\text{vorticity}} = \frac{1}{2} \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix}$$

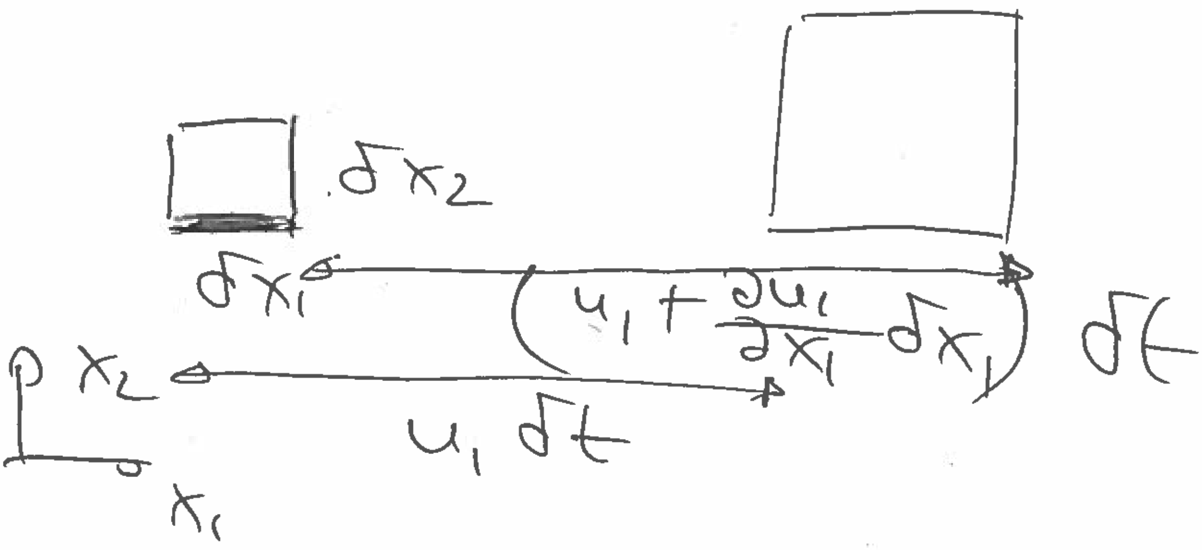
(2) The rate of strain tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

contains shearing & dilation
(stretching / compression)

(i) Extensional rate of strain

Illustration (2D)



$$\text{Strain} = \frac{\text{length} - \text{old length}}{\text{old length}}$$

$$= \frac{\left\{\left[\cancel{\partial x_1} + \left(\cancel{u_1} + \frac{\partial u_1}{\partial x_1}\cancel{\partial x_1}\right)\partial t\right] - \cancel{u_1 \partial t}\right\} - \cancel{\partial x_1}}{\cancel{\partial x_1}}$$

$$\text{Strain} = \frac{\partial u_1}{\partial x_1}\partial t$$

$$\text{rate of strain} = \frac{\partial \text{"strain"}}{\partial t}$$

$$= \frac{\partial u_1}{\partial x_1} = \varepsilon_{11}$$

Similar for other directions

$\boxed{\text{$\varepsilon_{11}$ (etc), i.e. the diag. entries of ε_{ij}, represent the extensional rate of strain in the direction of the cartesian coordinate axes}}$