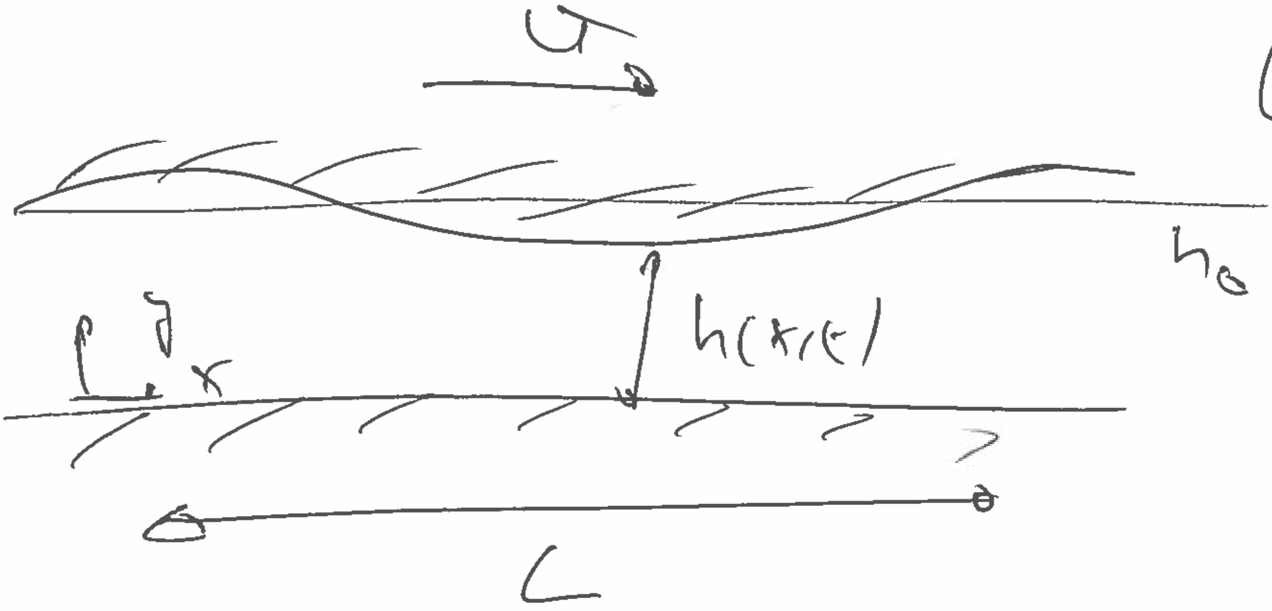
(1



$$\frac{h_0}{L} \ll 1$$

Scale:

$x = L\tilde{x}$ $\quad p = P\tilde{p}$

$y = h_0\tilde{y}$ $\quad t = \frac{L}{U}\tilde{t}$

$u = U\tilde{u}$

$v = V\tilde{v}$

Cont'n:

$$\frac{U}{L}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{V}{h_0}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\Rightarrow \quad V = U\frac{h_0}{L} \ll U.$$

x-mom. eqn:

$$\rho\left(\frac{U^2}{L}\frac{\partial \tilde u}{\partial \tilde t} + \frac{U^2}{L}\tilde u \frac{\partial \tilde u}{\partial \tilde x} + \frac{U^2 h_0}{L h_0}\tilde v\frac{\partial \tilde u}{\partial \tilde y}\right) =$$

$$= -\frac{P}{L}\frac{\partial \tilde p}{\partial \tilde x} + \mu\left(\frac{U}{L^2}\frac{\partial^2 \tilde u}{\partial \tilde x^2} + \frac{U}{h_0^2}\frac{\partial^2 \tilde u}{\partial \tilde y^2}\right)$$

$$\frac{\rho U^2}{L}\frac{D\tilde u}{D\tilde t} = -\frac{P}{L}\frac{\partial \tilde p}{\partial \tilde x} + \frac{\mu U}{h_0^2}\left(\left(\frac{h_0}{L}\right)^2\frac{\partial^2 \tilde u}{\partial \tilde x^2} + \frac{\partial^2 \tilde u}{\partial \tilde y^2}\right)$$

$$\cancel{\underbrace{\frac{\rho U h_0}{\mu}}_{Re}\underbrace{\left(\frac{h_0}{L}\right)}_{\ll 1}\frac{D\tilde u}{D\tilde t}} = -\frac{P}{\left(\frac{\mu U}{h_0}\right)}\left(\frac{h_0}{L}\right)\frac{\partial \tilde p}{\partial \tilde x} + \left(\cancel{\underbrace{\left(\frac{h_0}{L}\right)^2}_{\ll 1}\frac{\partial^2 \tilde u}{\partial \tilde x^2}} + \frac{\partial^2 \tilde u}{\partial \tilde y^2}\right)$$

can neglect $\frac{D\tilde u}{D\tilde t}$ if

$$Re\left(\frac{h_0}{L}\right) \ll 1$$

To retain a balance of forces between pressure & viscous effects choose

$$P = \frac{\mu U}{h_0} \frac{1}{\left(h_0/L\right)}$$

Note P is much larger than the typical shear stress $\frac{\mu U}{h_0}$

$$\boxed{0 = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}}$$

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y - comp. of mom. eqn:

(EXERCISE)

$$\boxed{\frac{\partial \tilde{p}}{\partial \tilde{y}} = 0}$$

Return to dimensional quantities:

$$\boxed{\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \\ 0 &= -\frac{\partial p}{\partial y} \end{aligned}}$$

These are the parallel flow eqn. $\Rightarrow$ The gentle slope of the upper wall means that locally the fluid behaves $\approx$ flow in a parallel-walled channel.

BC: $u(y=0) = 0$

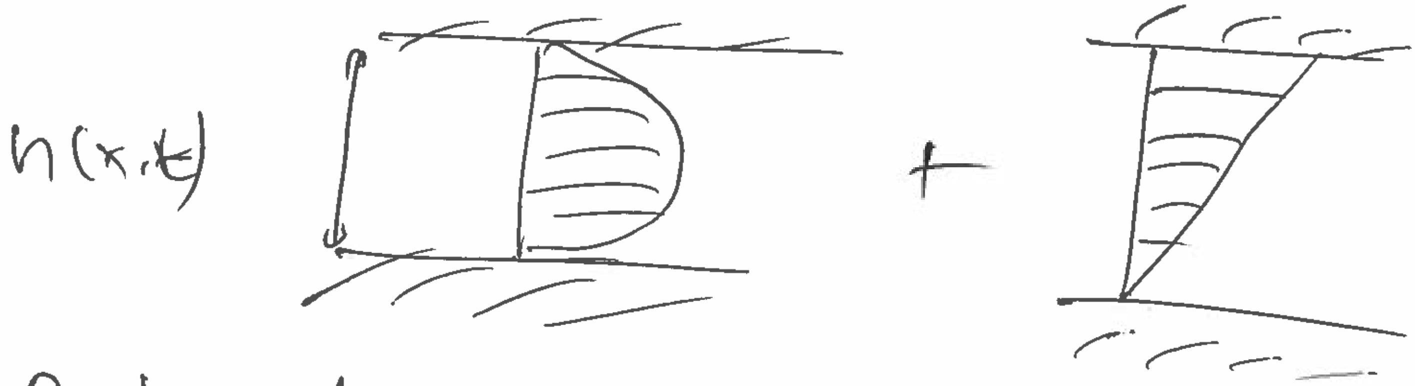
$u(y=h(x,t)) = U$

Since $\frac{\partial p}{\partial y} = 0 \Rightarrow$ pressure doesn't depend on y

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + Ay + B$$

A & B from BCs:

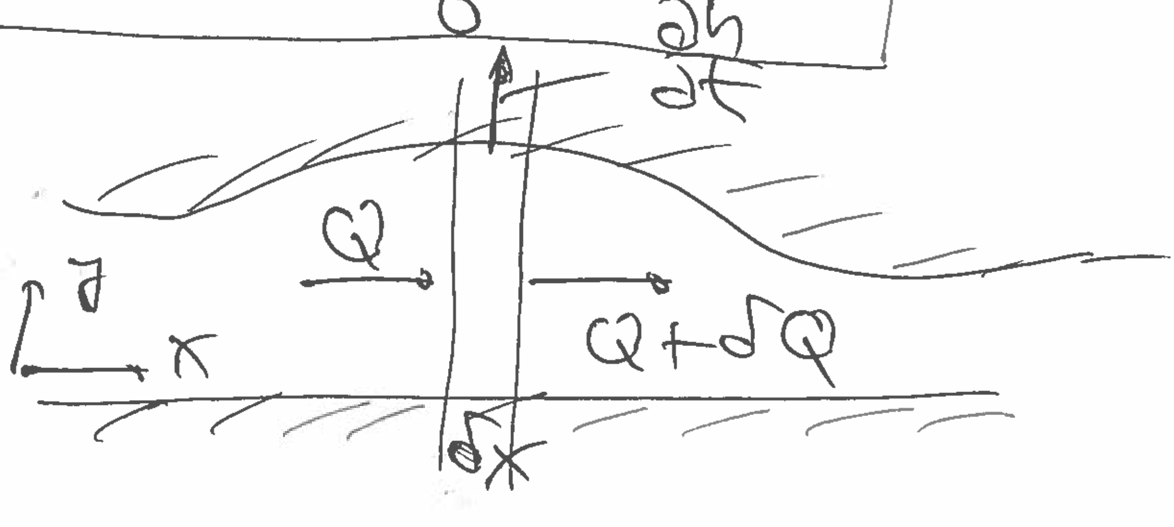
$$u(x,y,t) = \underbrace{\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h(x,t)\,y\right)} + U\frac{y}{h(x,t)}$$



But what is the pressure?
Addn requirement: volume conservation!

Consider volume flux:

$$\boxed{Q(x,t) = \int_0^{h(x,t)} u\,dy}$$



$$\delta Q + \frac{\partial h}{\partial t}\,\delta x = 0$$

(net outflow is zero)

$$\boxed{\frac{\partial Q}{\partial x} = -\frac{\partial h}{\partial t}}$$

Recall:

$$Q(x,t) = \int_0^{h(x,t)} u(x,y,t)\,dy$$

see above

$$Q(x,t) = -\frac{1}{12\mu}\frac{\partial p}{\partial x}h^3(x,t) + \frac{1}{2}U h(x,t)$$

$$\frac{\partial Q}{\partial x} = -\frac{1}{12\mu}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}h^3\right) + \frac{1}{2}U\frac{\partial h}{\partial x} = -\frac{\partial h}{\partial t}$$

$$\boxed{\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right) = 12\frac{\partial h}{\partial t} + 6U\frac{\partial h}{\partial x}}$$

Reynolds lubrication eqn.