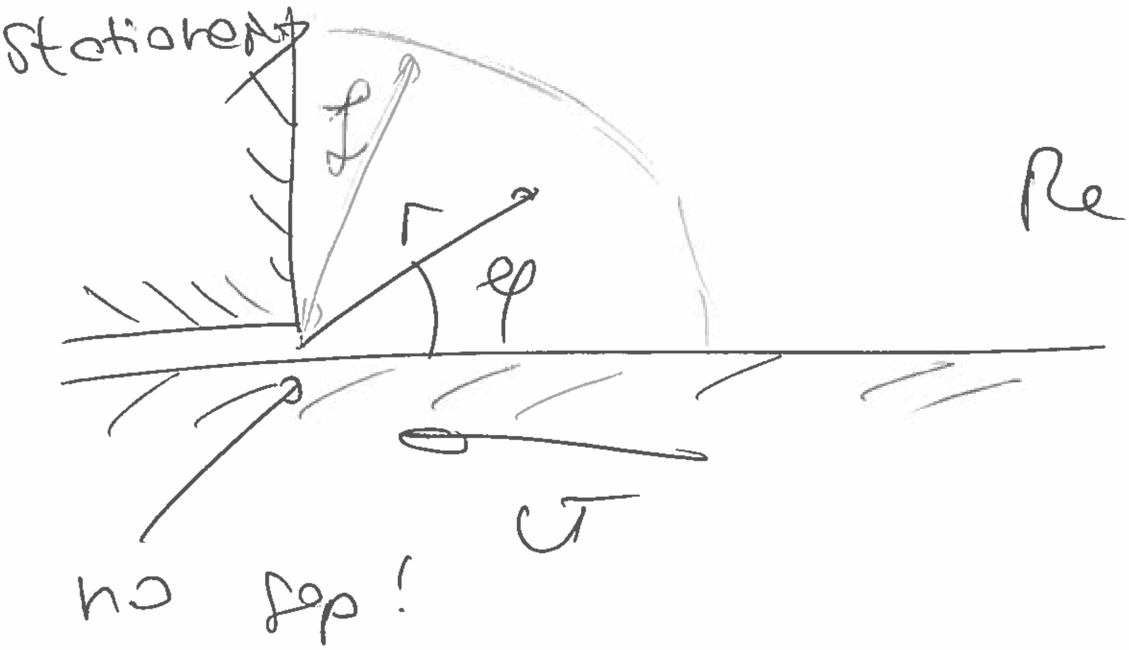
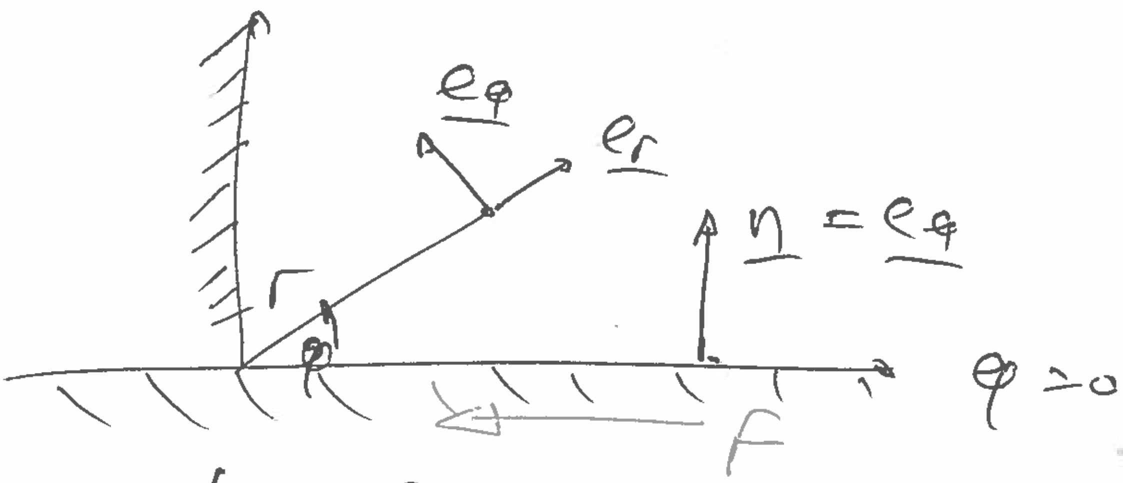


$$Re = \frac{U \rho \ell}{\mu}$$

$\underline{u}$ indep. of r !

(II) Traction on bottom wall



$$t_i = \tau_{ij} n_j$$

$$F = \left| \int_0^\infty \tau_{r\varphi} \, dr \right| \quad \text{(EXERCISE)}$$

$$\tau_{ij} = -p \delta_{ij} + 2\mu e_{ij} \qquad i,j = \{r, \varphi\}$$

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \qquad \epsilon_{\varphi\varphi} = \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{u}{r}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} \qquad \epsilon_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \varphi}\right]$$

$$\epsilon_{\varphi z} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial w}{\partial \varphi} + \frac{\partial v}{\partial z}\right] \qquad \epsilon_{rz} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right]$$

$$\tau_{r\varphi} = 2\mu e_{r\varphi}$$

$$\frac{\tau_{r\varphi}}{\mu} = r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \varphi}$$

at $\varphi = 0$

Recall u, v indep of r

$$r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) \sim r\, r^{-2} \sim \frac{1}{r}$$

$$\frac{1}{r}\frac{\partial u}{\partial \varphi} \sim \frac{1}{r}$$

$$F = \mu \int_0^\infty \cdots \frac{1}{r}\, dr \to \infty$$

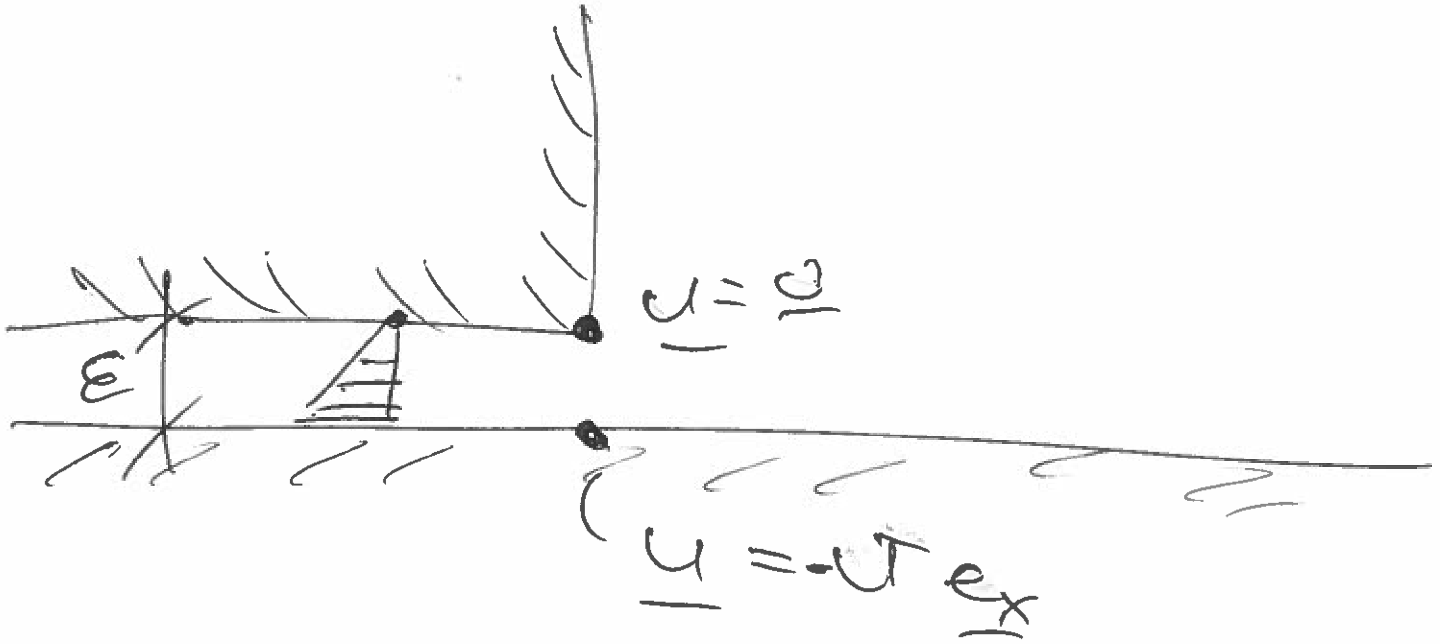
diverges

Do we need an ∞ force to scrape fluid off plate?

Admittedly we're integrating over an ∞ large plate.

But the singularity also arises near $r=0$.

$\Rightarrow$ Trouble arises at corner!



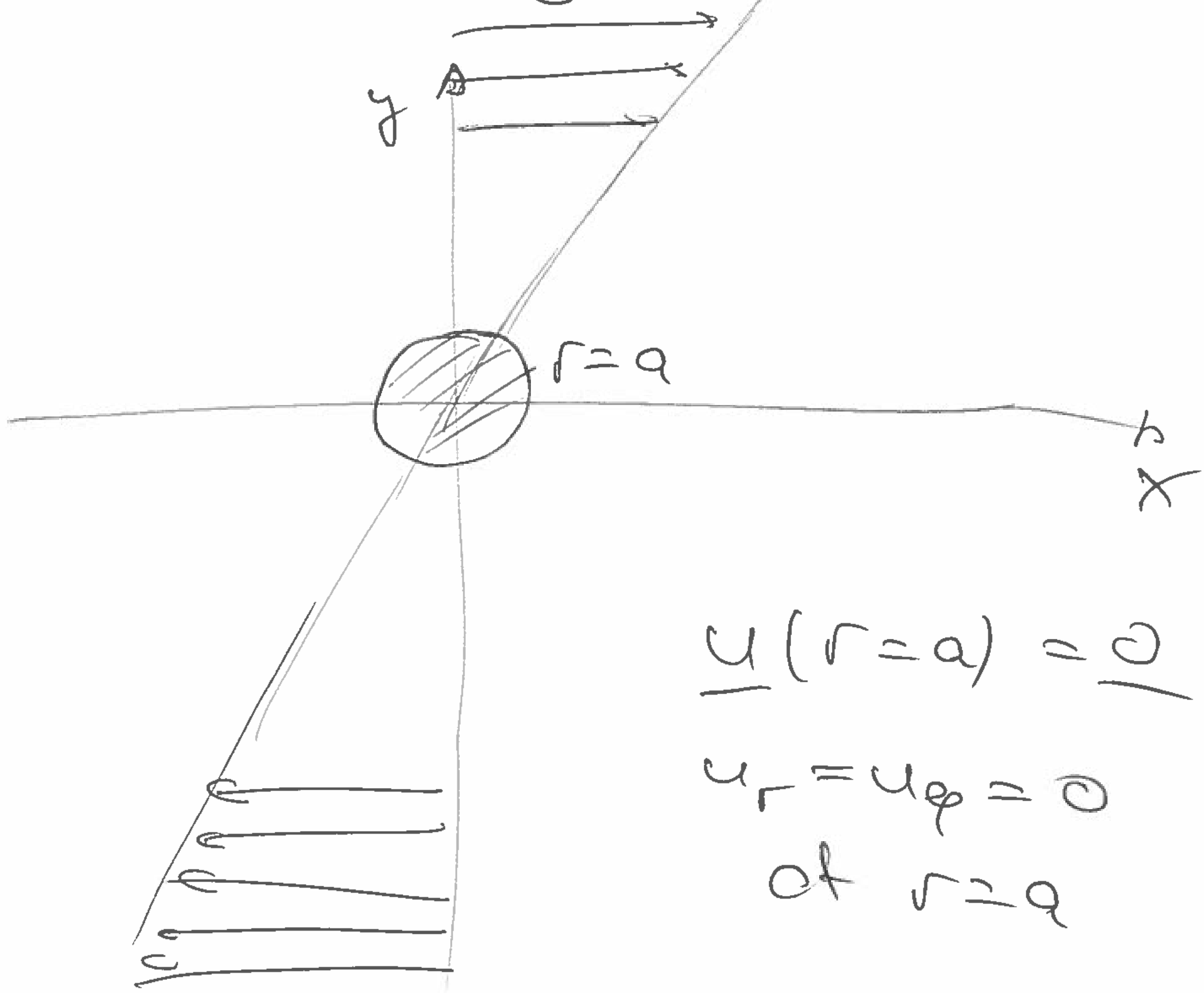
$$e_{r\varphi} \sim e_{xy} \sim \frac{U}{\varepsilon} \to \infty \text{ as } \varepsilon \to 0$$

The jump in velocity over a zero distance causes a non-integrable stress on the wall!

This is a modelling problem!

- continuum hypothesis breaks down
- could fix this by allowing a small gap $\Rightarrow$ problem too hard.

Example: Cylinder in a shear flow



$\underline{u}(r=a) = \underline{0}$

$u_r = u_\varphi = 0$ at $r=a$

In far field:

$\underline{u} \to \gamma y \underline{e_x}$ as $r \to \infty$

constant

Polars:

$y = r\sin\varphi$

$\underline{e_x} = \underline{e_r}\cos\varphi - \underline{e_\varphi}\sin\varphi$

8.2.1 The streamfunction and the biharmonic equation in cylindrical polars

- In cylindrical polars, (r, φ) the relation between the streamfunction $\psi(r, \varphi)$ and the velocity components u_r and u_φ is:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \tag{8.14}$$

and

$$u_\varphi = -\frac{\partial \psi}{\partial r}, \tag{8.15}$$

where $\mathbf{u} = u_r \mathbf{e}_r + u_\varphi \mathbf{e}_\varphi$.

- The biharmonic equation in polar coordinates:

$$\nabla^4 \psi(r, \varphi) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right]\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \varphi^2}\right] \tag{8.16}$$

$$\nabla^4 \psi(r, \varphi) = \psi_{,rrrr} + \frac{2}{r}\psi_{,rrr} - \frac{1}{r^2}(\psi_{,rr} - 2\psi_{,rr\varphi\varphi}) + \frac{1}{r^3}(\psi_{,r} - 2\psi_{,r\varphi\varphi}) + \frac{1}{r^4}(4\psi_{,\varphi\varphi} + 2\psi_{,\varphi\varphi\varphi\varphi}) \tag{8.17}$$

- For axisymmetric solutions:

$$\nabla^4 \psi(r) = \frac{1}{r}\left[r\left(\frac{1}{r}[r\psi_{,r}]_{,r}\right)_{,r}\right]_{,r} \tag{8.18}$$

$$\nabla^4 \psi(r) = \psi_{,rrrr} + \frac{2}{r}\psi_{,rrr} - \frac{1}{r^2}\psi_{,rr} + \frac{1}{r^3}\psi_{,r} \tag{8.19}$$

- The general form of the solution of the biharmonic equation in cylindrical polars is known. It can be represented by superposition of the following solutions:
  - The general axisymmetric solution:

$$\psi(r) = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r \tag{8.20}$$

  - The general separated non-axisymmetric solution:

    For $n = 1$:

$$\psi(r, \varphi) = \left(Ar + \frac{B}{r} + Cr^3 + Dr\ln r\right) \cos(\varphi) + \left(ar + \frac{b}{r} + cr^3 + dr\ln r\right) \sin(\varphi) \tag{8.21}$$

    For $n \geq 2$:

$$\psi(r, \varphi) = \sum_{n=2}^{\infty} \left(A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}\right) \cos(n\varphi) + \left(a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{-n+2}\right) \cdot \sin(n\varphi) \tag{8.22}$$

The coefficients $(A_0, B_0, C_0, D_0, A_1, B_1, C_1, D_1, a_1, b_1, c_1, d_1, A_2, B_2, C_2, D_2, a_2, b_2, c_2, d_2, \ldots)$ have to be determined from the boundary conditions.

$$\underline{u} \to S\,\underbrace{r\sin\varphi}_{y}\left(\cos\varphi\,\underline{e_r} - \sin\varphi\,\underline{e_\varphi}\right)$$

$$\underline{u} \longrightarrow S\,r\left(-\underbrace{\sin^2\varphi}_{\frac{1}{2}(1-\cos(2\varphi))}\,\underline{e_\varphi} + \underbrace{\sin\varphi\cos\varphi}_{\frac{1}{2}\sin(2\varphi)}\,\underline{e_r}\right)$$

as $r \to \infty$

$$\left.\begin{aligned} u_r &\to \tfrac{1}{2} S r \sin(2\varphi) \\ u_\varphi &\to -\tfrac{1}{2} S r \left(1 - \cos(2\varphi)\right) \end{aligned}\right\} \text{ as } r \to \infty$$

$$\nabla^4 \psi = 0$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \qquad u_\varphi = -\frac{\partial \psi}{\partial r}$$

We use the general separated solution of $\nabla^4 \psi(r, \varphi)$ from the handout.
Only include terms that are req'd by the BCs.