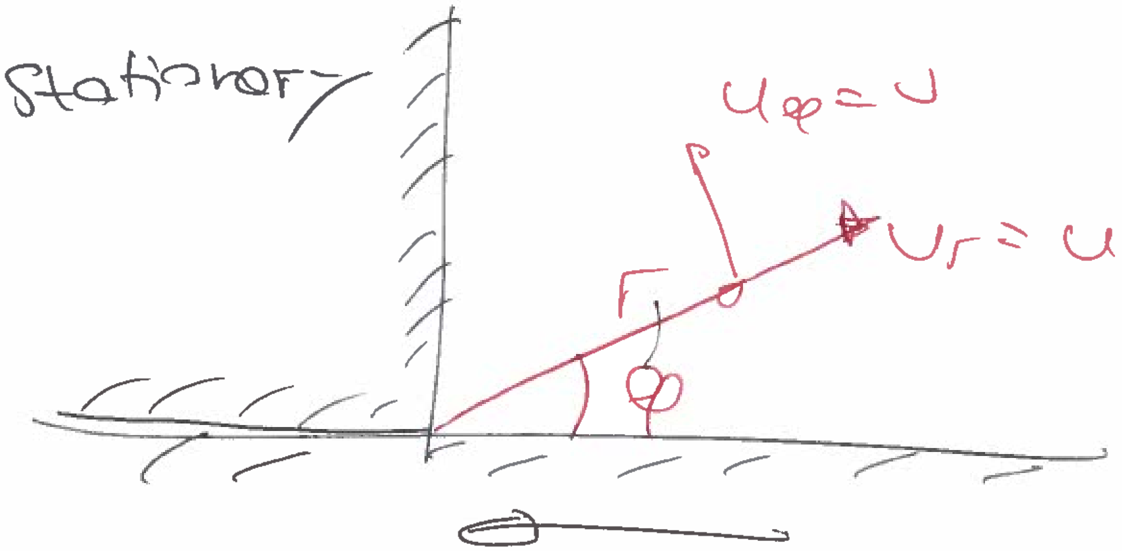
1



$$\boxed{\nabla^4 \psi = 0}$$

$Re \ll 1$

<u>No penetration:</u>

$$u_\varphi = -\frac{\partial \psi}{\partial r} = 0 \quad @ \ \varphi = 0 \ \& \ \pi/2$$

$$\Rightarrow \boxed{\begin{aligned} &\psi(\varphi = 0) = 0 \quad (1) \\ &\psi(\varphi = \tfrac{\pi}{2}) = 0 \quad (2) \end{aligned}}$$

<u>No slip:</u>

$$u = \boxed{0 = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad @ \ \varphi = \frac{\pi}{2}} \quad (3)$$

$$u = \boxed{-U = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad @ \ \varphi = 0} \quad (4)$$

<u>Try</u> separated soln:

$$\psi(r,\varphi) = g(r)\, f(\varphi)$$

<u>Note:</u> BCs (3) & (4) show that

$\frac{1}{r}\frac{\partial \psi}{\partial \varphi}$ should be independent of r for fixed φ

<u>Try:</u>

$$g(r) = U r$$

<u>Ansatz:</u> $\boxed{\psi(r,\varphi) = U r f(\varphi)}$

Into BCs:

$\varphi = 0$: $\psi = 0$:

$\varphi = \frac{\pi}{2}$ $\psi = 0$

$\varphi = 0$: $\frac{1}{r}\frac{d\psi}{\partial \varphi} = -U$

$\varphi = \frac{\pi}{2}$ $\frac{1}{r}\frac{d\psi}{\partial \varphi} = 0$

$$\boxed{\begin{aligned} f(0) &= 0 \\ f\left(\tfrac{\pi}{2}\right) &= 0 \\ f'(0) &= -1 \\ f'\left(\tfrac{\pi}{2}\right) &= 0 \end{aligned}}$$

PDE $\nabla^4 \psi = \nabla^2 \nabla^2 \psi = 0$

$$\nabla^2 \psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \right) U r \cdot f(\varphi)$$

$$= \frac{U}{r} f + \frac{U}{r} f'' = U r^{-1} (f + f'')$$

$$\nabla^4 \psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \right) U r^{-1} (f + f'')$$

$$= U \Big\{ (-1)(-2) r^{-3} (\underline{\underline{f}} + \underline{\underline{f''}}) + (-1) r^{-3} (\underline{\underline{f}} + \underline{\underline{f''}})$$

$$+ r^{-3} (\underline{\underline{f''}} + f^{IV}) \Big\}$$

$$= \frac{U}{r^3} \Big\{ f(2-1) + f''(2-1+1) + f^{IV} \Big\}$$

$$\nabla^4 \psi = \frac{U}{r^3} \underbrace{\left(f + 2f'' + f^{IV} \right)}_{=0} \overset{!}{=} 0$$

ODE for $f(\varphi)$ 4^{th} order const. coeff

$$f(\varphi) \sim e^{\lambda \varphi}$$

char. poly.:

$$1 + 2\lambda^2 + \lambda^4 = 0$$

$$(1+\lambda^2)^2 = 0$$

$$\lambda_{1234} = \pm i$$

$$f(\varphi) = A\sin\varphi + B\cos\varphi + C\varphi\sin\varphi + D\varphi\cos\varphi$$

Apply BCs to determine A, B, C, D

...

$$\boxed{\psi(r,\varphi) = \frac{Ur}{\left(\frac{\pi}{2}\right)^2 - 1}\left(-\left(\frac{\pi}{2}\right)^2\sin\varphi + \varphi\cos\varphi + \frac{\pi}{2}\varphi\sin\varphi\right)}$$

So

$$\left.\begin{aligned} u_\varphi &= -\frac{\partial\psi}{\partial r} \\ u_r &= \frac{1}{r}\frac{\partial\psi}{\partial\varphi} \end{aligned}\right\} \text{ indep of } r$$

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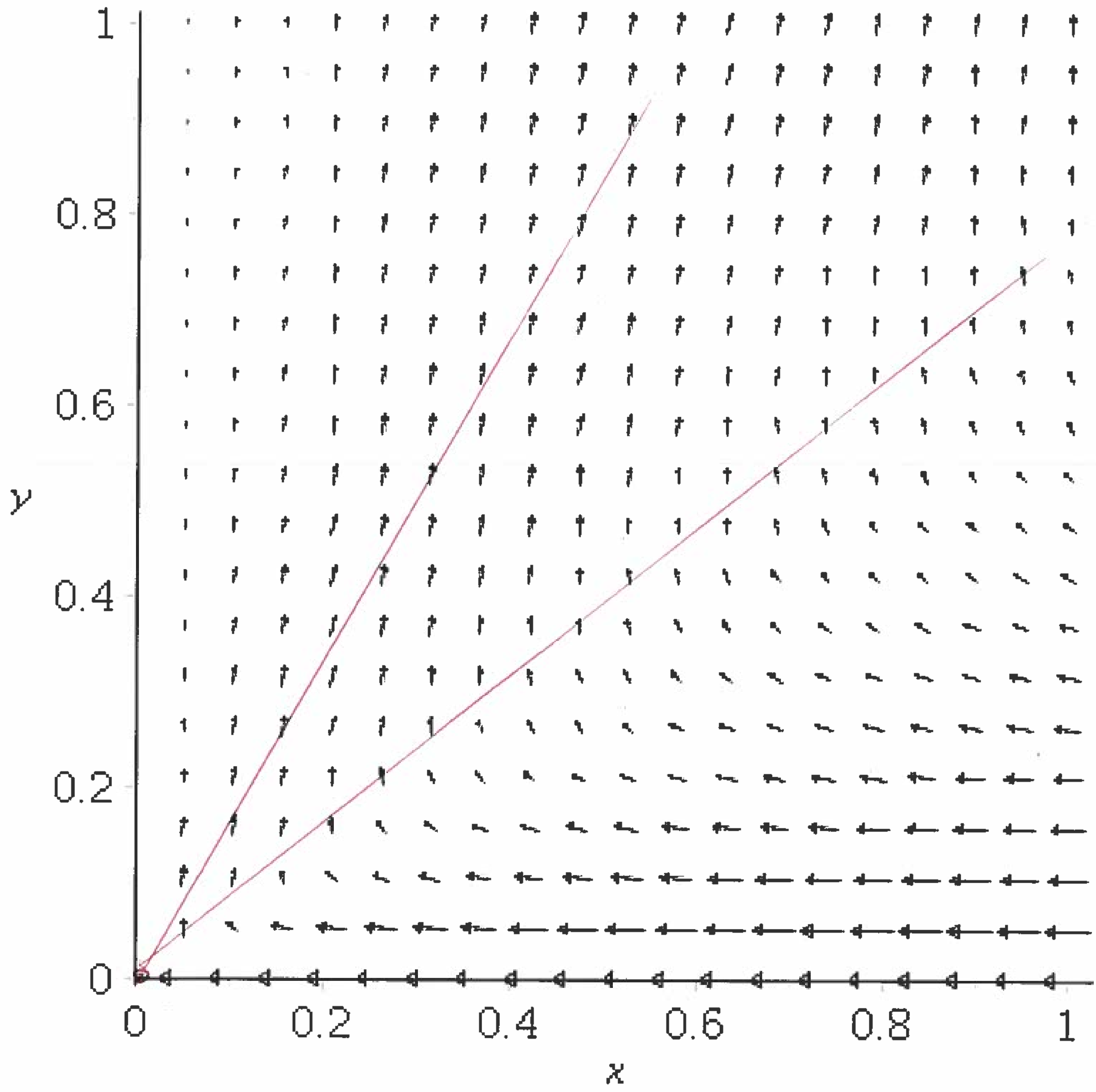


Discussion

(I) Non-uniformity of the soln.

We had assumed $Re \ll 1$

$$Re = \frac{U\,l\,\rho}{\mu}$$

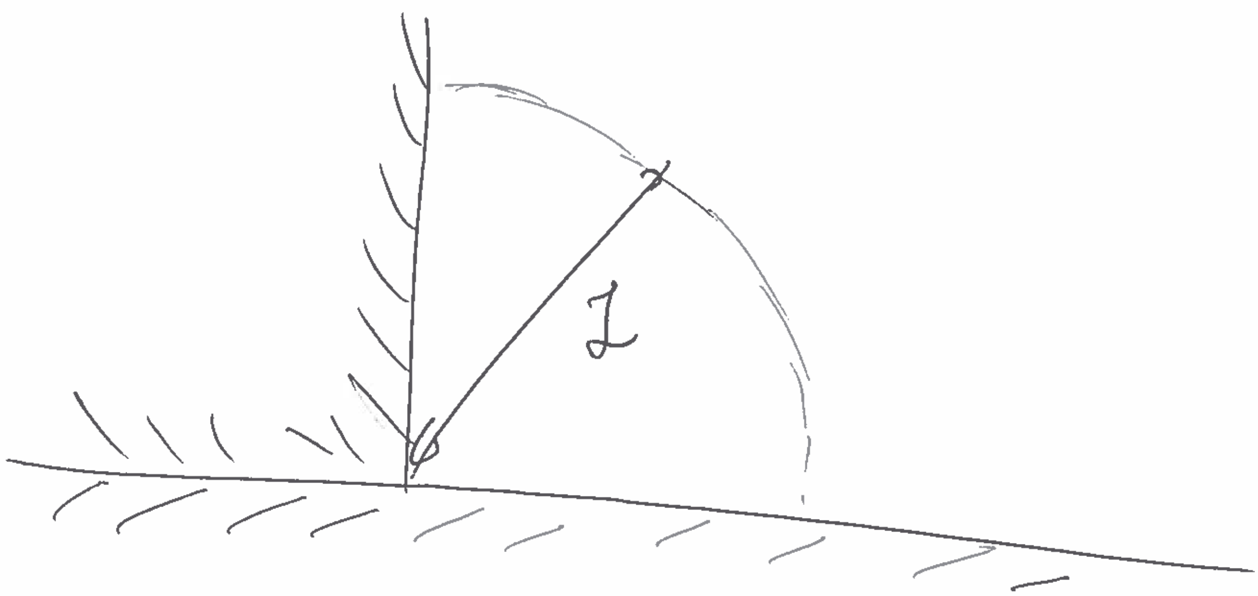
veloc. of bottom wall

fluid properties

What is l?

Problem does not have a lengthscale!

In order to define a lengthscale we have to choose one:



With this choice (distance l from corner) we can choose a veloc, say, for which

$$Re = \frac{U l \rho}{\mu} \ll 1$$

Alternatively: Given U, ρ, μ we can determine how close we have to be to corner to make Stokes eqns valid (= good approx. to N.St)

<u>But:</u> At large distances from corner Stokes eqns are <u>not</u> valid!

Stokes eqns provide a local soln which is not uniformly valid in entire domain.