Fluid mechanics

3 steps: I Describe mathematically the flow field / motion of fluid particles: Kinematics

II Formulate the eqns. of motion: Balance of forces acting on fluid particles: Stresses

III Constitutive eqns: relate the Kinematics to stresses

} The Navier-Stokes eqns.

Then: lots of examples!

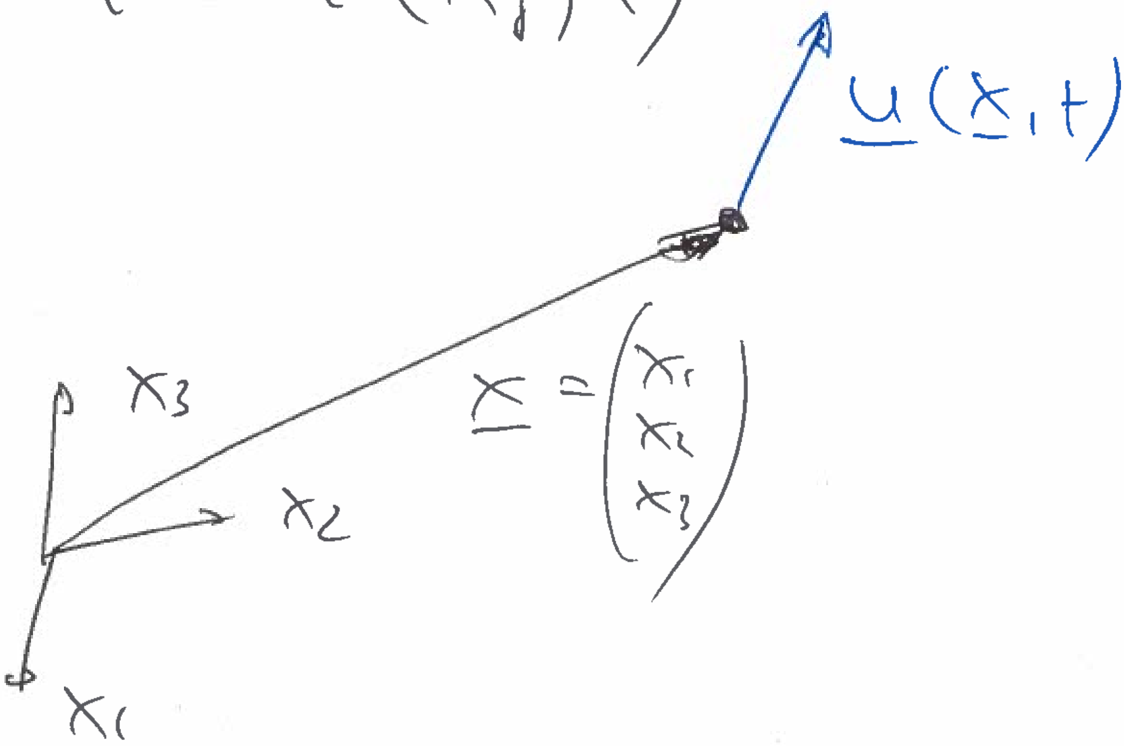
§2 Kinematics

The Eulerian flow field

Assume we know the velocity $\underline{u}$ as a fct. of the three cartesian coordinates (x_1, x_2, x_3) & time t.

$$\underline{u} = \underline{u}(x_1, x_2, x_3, t) = \underline{u}(\underline{x}, t)$$

$$u_i = u_i(x_j, t)$$



At time t the fluid (3)
particle at this position ($\underline{x}$)
has velocity $\underline{u}(\underline{x},t)$

<u>Note:</u> At different times, different material particles occupy position $\underline{x}$!
(Eulerian vs. Lagrangian viewpoint):
This has important implications

<u>Example:</u> Acceleration of fluid particles.

The material derivative

The position $\underline{x}$ of a fluid particle is given by

$$\underline{x} = \underline{x}^p(t) = \begin{pmatrix} x_1^p(t) \\ x_2^p(t) \\ x_3^p(t) \end{pmatrix}$$

(Trajectory)

velocity of the particle is

$$\underline{u} = \underline{u}\left(x_1^p(t),\, x_2^p(t),\, x_3^p(t),\, t\right)$$

So, to get the acceleration of that particle:

$$\frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t} + \left.\frac{\partial \underline{u}}{\partial x_1}\right|_p \frac{dx_1^p}{dt} + \left.\frac{\partial \underline{u}}{\partial x_2}\right|_p \frac{dx_2^p}{dt} + \left.\frac{\partial \underline{u}}{\partial x_3}\right|_p \frac{dx_3^p}{dt}$$

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \left.\frac{\partial u_i}{\partial x_j}\right|_p \underbrace{\frac{dx_j^p}{dt}}_{u_j}$$

$$\boxed{\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}}$$

Symbolically:

$$\frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}$$

This expression is often denoted by

$$\frac{D\underline{u}}{Dt}$$

to distinguish derivs. assoc. with an individual material particle from the deriv. at a fixed spatial point.

The rate of strain tensor & the vorticity

Velocity field in itself is not very interesting.

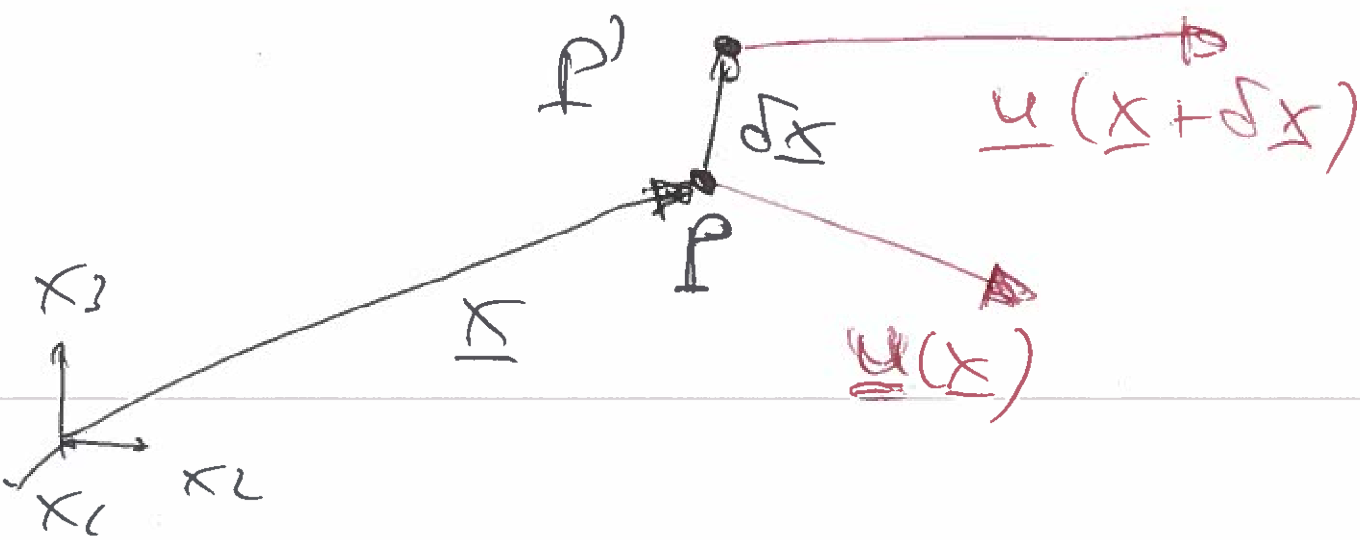
It contains:

- translations
- rotations
- shearing
- dilation

$\Big\}$ motions

How do we identify these?

Examine the veloc. field in vicinity of point P.



Taylor expand:

$$\underline{u}(\underline{x}+\delta\underline{x}) = \underline{u}(x_1+\delta x_1, x_2+\delta x_2, x_3+\delta x_3)$$

$$= \underline{u}(x_1, x_2, x_3) + \underbrace{\frac{\partial \underline{u}}{\partial x_1}\delta x_1 + \frac{\partial \underline{u}}{\partial x_2}\delta x_2 + \frac{\partial \underline{u}}{\partial x_3}\delta x_3}_{\delta \underline{u}} + O(\delta x_i^2)$$

$$\delta \underline{u} = \underline{u}(\underline{x}+\delta\underline{x}) - \underline{u}(\underline{x})$$

$$\delta u_i = \frac{\partial u_i}{\partial x_j}\delta x_j + O(\delta x_k^2)$$

Now $\delta x_j \to dx_j$

$$du_i = \underbrace{\frac{\partial u_i}{\partial x_j}}_{\text{"velocity gradient tensor" } 3\times3 \text{ matrix}} dx_j$$