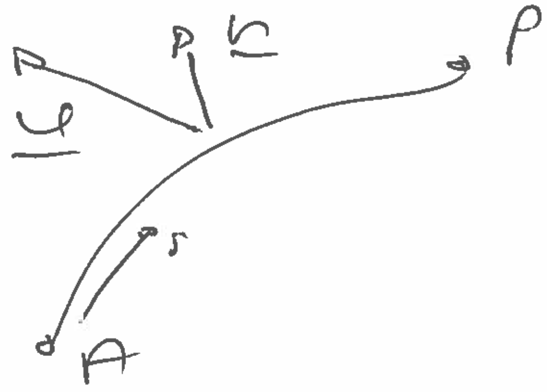
$$\psi_A(\underline{P}) = \int_A^P \underline{u}\cdot\underline{n}\, ds$$

(x, y)



$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x} \qquad \nabla\cdot\underline{u} = 0$$

$$\omega = -\nabla^2\psi$$

$$\underline{\omega} = \text{curl}\,\underline{u} = \omega\,\underline{e_z}$$

Stream fcn & vorticity eqn in 2D:

$$\boxed{\begin{gathered} \frac{D\omega}{Dt} = \nu\nabla^2\omega \\ \omega = -\nabla^2\psi \\ u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x} \end{gathered}}$$

The Vorticity equation

- Use of the streamfunction allows us to automatically satisfy the continuity equation. Now we will try to transform the momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

into an equation for the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

- Here are a few results that we will use in the derivation:

$$\frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) = (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}), \tag{1}$$

$$\nabla \times \nabla \phi = 0, \tag{2}$$

and

$$\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} + \mathbf{u}\underbrace{\nabla \cdot \boldsymbol{\omega}}_{0} - \boldsymbol{\omega}\underbrace{\nabla \cdot \mathbf{u}}_{0}, \tag{3}$$

where the last two terms vanish because

$$\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \mathbf{u}) = \operatorname{div}\operatorname{curl}\mathbf{u} = 0$$

and

$$\nabla \cdot \mathbf{u} = 0.$$

- First, we use (1) in the momentum equation to obtain

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times\underbrace{(\nabla\times\mathbf{u})}_{\boldsymbol{\omega}} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}$$

i.e.

$$\underbrace{\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times\boldsymbol{\omega}}_{LHS} = \underbrace{-\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}}_{RHS}$$

- Now take the curl of the LHS:

$$\nabla\times LHS = \frac{\partial}{\partial t}\underbrace{(\nabla\times\mathbf{u})}_{\boldsymbol{\omega}} + \frac{1}{2}\underbrace{\nabla\times\nabla(\mathbf{u}\cdot\mathbf{u})}_{0\text{ because of (2)}} - \underbrace{\nabla\times(\mathbf{u}\times\boldsymbol{\omega})}_{\text{see (3)}}$$

i.e.

$$\nabla\times LHS = \frac{\partial\boldsymbol{\omega}}{\partial t} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$$

- ...and the RHS:

$$\nabla\times RHS = -\frac{1}{\rho}\underbrace{\nabla\times\nabla p}_{0\text{ because of (2)}} + \nu\nabla^2\underbrace{(\nabla\times\mathbf{u})}_{\boldsymbol{\omega}}$$

- Now combine the remaining non-zero terms

$$\underbrace{\frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega}}_{D\boldsymbol{\omega}/Dt} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} = \nu\nabla^2\boldsymbol{\omega}.$$

- The resulting equation is the *vorticity transport equation*

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega} \tag{4}$$

  which shows that the rate of change of the vorticity of material particles, $D\boldsymbol{\omega}/Dt$, is controlled by 'vortex stretching' (described by $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$; this is a familiar result from inviscid fluid mechanics) and by diffusion (described by $\nu\nabla^2\boldsymbol{\omega}$). The diffusion of vorticity only occurs in viscous flows.

- For 3D flows, the first term on the RHS in (4) represents vortex stretching: velocity gradients lead to a change in the rate of rotation of material particles.

- Note that for 2D flows, vortex stretching is absent since $\mathbf{u} = u(x, y)\ \mathbf{e}_x + v(x, y)\ \mathbf{e}_y$ and $\boldsymbol{\omega} = \omega(x, y)\ \mathbf{e}_z$ and therefore $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = 0$.

- The vorticity transport equation provides an interesting interpretation of the kinematic viscosity ν: The kinematic viscosity is the diffusion coefficient for the diffusion of vorticity.

- Many phenomena in viscous fluid mechanics can be interpreted in terms of the diffusion of vorticity but this is (unfortunately) beyond the scope of this course.

- For 2D flows, the vorticity transport equation

$$\frac{D\omega}{Dt} = \nu\nabla^2\omega$$

together with the equation for the vorticity in terms of the streamfunction

$$\omega = -\nabla^2\psi$$

and

$$u = \partial\psi/\partial y \quad \text{and} \quad v = -\partial\psi/\partial x$$

provide the streamfunction-vorticity formulation of the Navier-Stokes equations, which consists of only two PDEs for the scalars ω and ψ rather than the three equations for u, v and p in the 'primitive variable' form.

- Scaling arguments show that in the limit of zero Reynolds number, only one fourth-order PDE for the streamfunction ψ needs to be solved, namely the biharmonic equation

$$\nabla^4\psi = 0,$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}.$$

- This can also be shown directly by taking the curl of the Stokes equations.

Can nondim. these eqns & consider the limit $Re \to 0$.

Alternative: Take curl of the Stokes eqns (2D):

$$\cancel{Re \frac{D\tilde{\underline{u}}}{D\tilde{t}}} = -\nabla\tilde{p} + \nabla^2\tilde{\underline{u}}$$

$$\underbrace{\nabla\times\nabla\tilde{p}}_{} = \nabla\times\nabla^2\tilde{\underline{u}}$$

$$\underline{0} = \nabla^2\,\nabla\times\tilde{\underline{u}} = \nabla^2\tilde{\underline{\omega}} = \underline{0}$$

2D: $\tilde{\underline{\omega}} = \tilde{\omega}_z\,\underline{e_z} = \tilde{\omega}\,\underline{e_z}$

$$\tilde{\omega} = -\nabla^2\tilde{\psi}$$

$$-\nabla^2\nabla^2\tilde{\psi} = 0$$

Back to dimensional variables

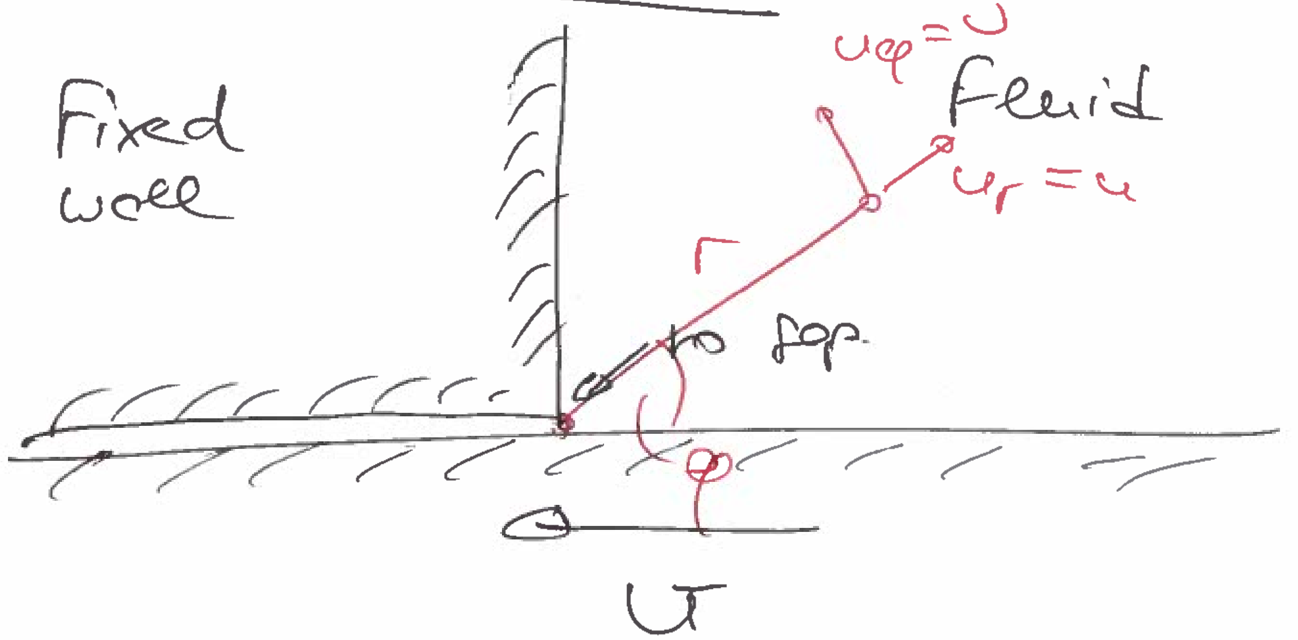
$$\boxed{\nabla^2\nabla^2\psi = \nabla^4\psi = 0}$$

$$\boxed{\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)\psi = 0}$$

One single 4th order PDE for $\psi(x,y)$ instead of 3 eqns for u, v, p.

Stokes-flow example

Scraping flow



Assume: · slow, steady flow
· very viscous fluid

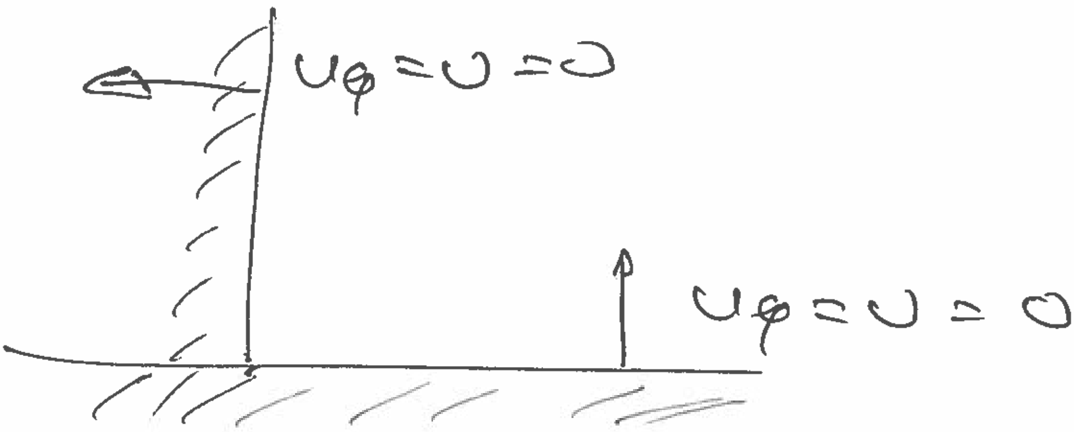
$\Rightarrow$ Stokes eqns apply since $Re \ll 1$ $\quad D^4\psi = 0$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

$$u = u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi}$$

$$v = u_\varphi = -\frac{\partial \psi}{\partial r}$$

BC: Impermeability



So:

$v = -\frac{\partial \psi}{\partial r} = 0$ at $\varphi = 0$ $\forall r$

$\psi = \text{const} = C_1$ at $\varphi = 0$

$v = -\frac{\partial \psi}{\partial r} = 0$ at $\varphi = \frac{\pi}{2}$ $\forall r$

$\psi = \text{const.} = C_2$ at $\varphi = \frac{\pi}{2}$

Continuity of streamfct across corner requires

$$C_1 = C_2 = C$$

because no flow into gap!

Constant is arbitrary so choose $C = 0$.