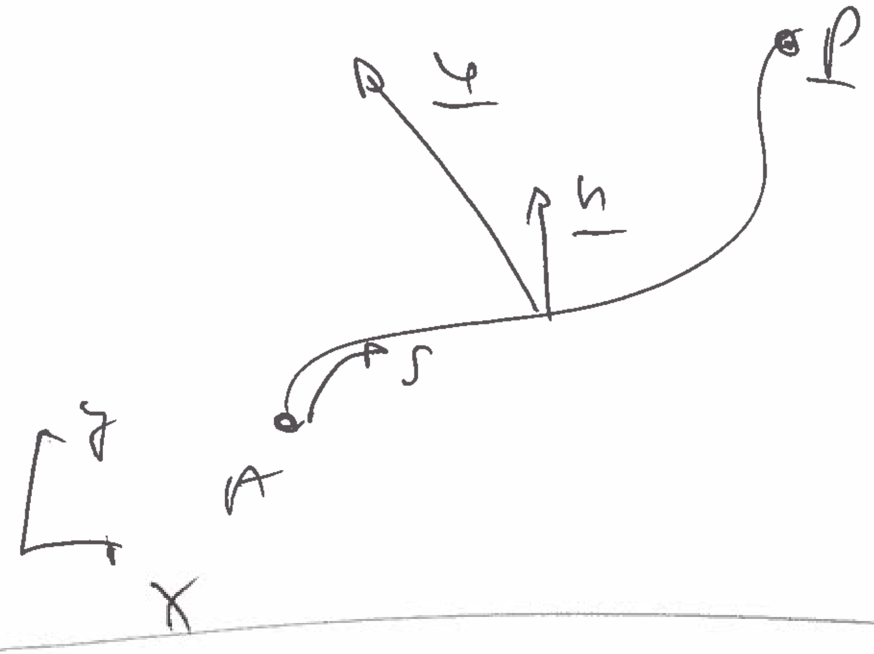
(1

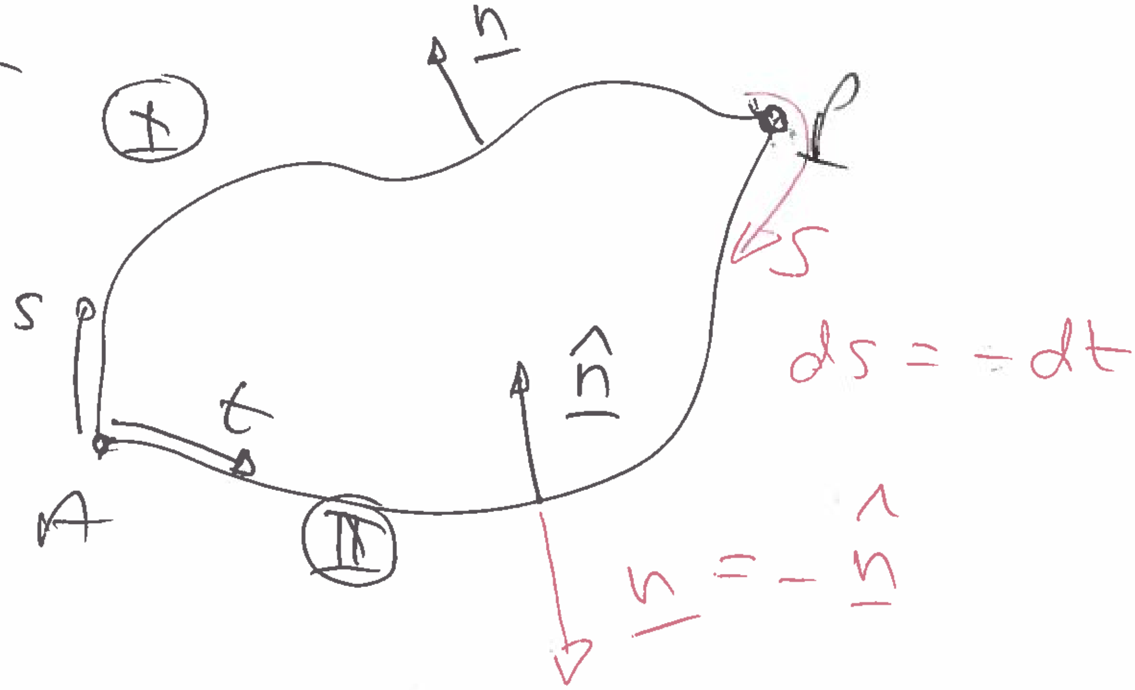
$$\psi_A(\underline{P}) = \int_A^P \underline{u} \cdot \underline{n} \, ds$$



Implications

(1) $\psi_A(\underline{P})$ is path independent

Proof:



$$\psi_A^{\text{I}}(P) = \int_{A\,(\text{I})}^{P} \underline{u}\cdot\underline{n}\,ds$$

$$\psi_A^{\text{II}}(P) = \int_{A\,(\text{II})}^{P} \underline{u}\cdot\underbrace{\hat{\underline{n}}}_{(-\underline{n})}\,\underbrace{d\hat{s}}_{(-ds)}$$

$$= -\int_{P\,(\text{II})}^{A} \underline{u}\cdot\underline{n}\,ds$$

<u>So:</u>

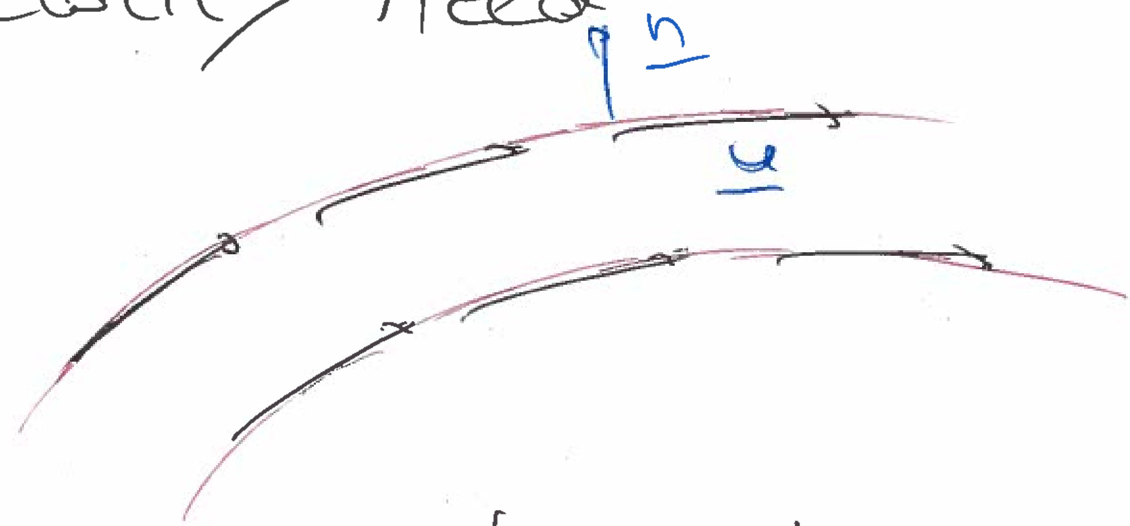
$$\psi_A^{\text{I}}(P) - \psi_A^{\text{II}}(P) = \int_{A\,(\text{I})}^{P} \underline{u}\cdot\underline{n}\,ds + \int_{P\,(\text{II})}^{A} \underline{u}\cdot\underline{n}\,ds$$

$$= \oint_{APA} (\underline{u}\cdot\underline{n})\,ds = 0$$

because flow is incompressible.

q.e.d.

(2) ψ is constant along streamlines. Obvious because ~~along~~ streamlines are tangentiable to the velocity field



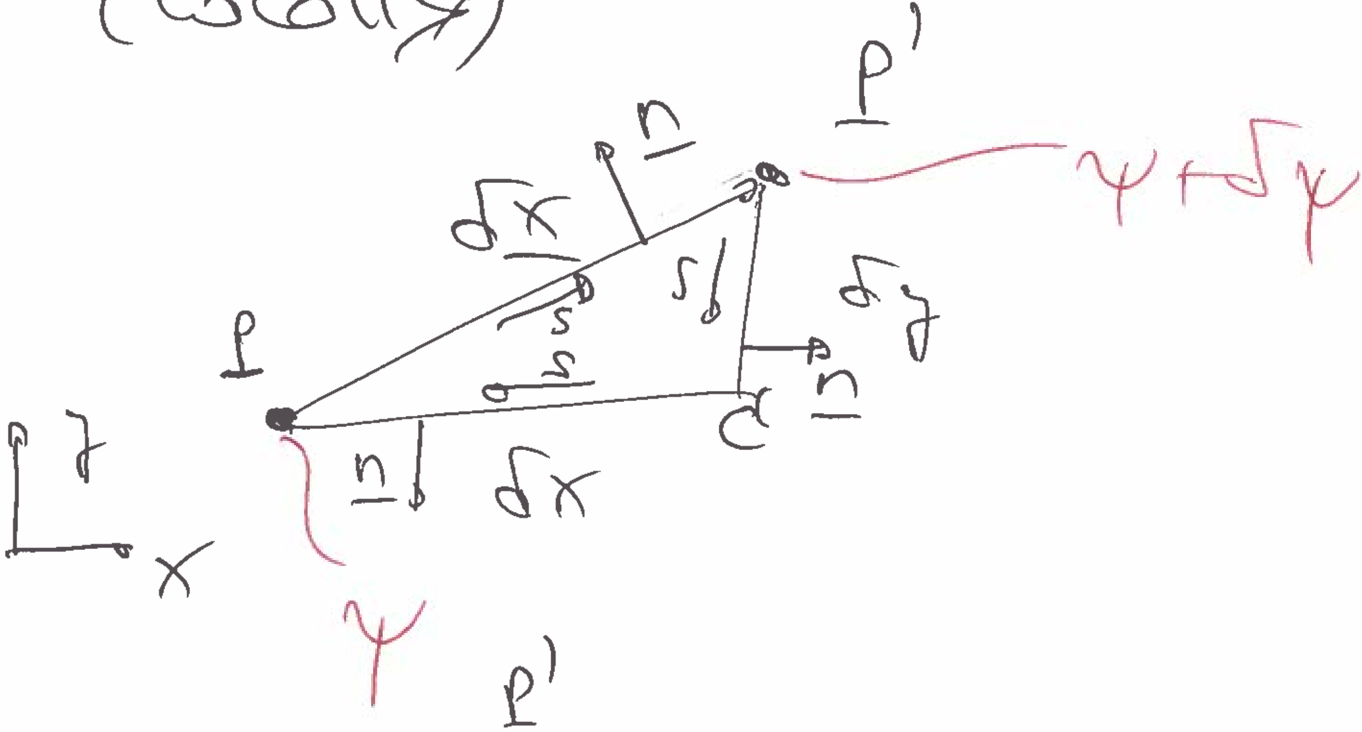
$\underline{u} \cdot \underline{n} = 0$ along streamlines

(3) impermeable boundaries are also streamlines:

$$\underline{u} \cdot \underline{n} = 0$$

<u>Convention:</u> set $\psi = 0$ along solid boundaries.

What is the relation between ψ & $\underline{u}$? (locally)



$$\delta\psi = \int_{P}^{P'} \underline{u}\cdot\underline{n}\, ds$$

Again: continuity:

$$\oint_{\triangle} \underline{u}\cdot\underline{n}\, ds = 0$$

$$\delta\psi = \int_{P}^{P'} \underline{u}\cdot\underline{n}\, ds = -\int_{P'}^{C} \underbrace{\underline{u}}_{u}\cdot\underbrace{\underline{n}\, ds}_{(-dy)} - \int_{C}^{P} \underbrace{\underline{u}}_{(-v)}\cdot\underbrace{\underline{n}\, ds}_{(-dx)}$$

$\delta x, \delta y \to 0$ + Mean value theorem

$$\delta\psi = u\,\delta y - v\,\delta x$$

Can also regard $\psi(x,y)$
2D Taylor expansion

$$\delta\psi = \frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y$$

$$\boxed{\begin{aligned} \frac{\partial\psi}{\partial x} &= -v \\ \frac{\partial\psi}{\partial y} &= u \end{aligned}}$$

Similar to potential, Airy stress fct. etc.

Remarks:

(1) Derivation involved continuity eqn.
This eqn. shoued be satisfied automatically!
Is it?

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \overset{?}{=} 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \overset{?}{=} 0 \quad \checkmark$$

$\Rightarrow$ If we formulate the problem in terms of ψ we can ignore the continuity eqn

(2) Relation to vorticity (2D)

$$\underline{\omega} = \omega \, \underline{e_z}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\omega = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$$

$$\boxed{\omega = -\nabla^2 \psi}$$

in 2D