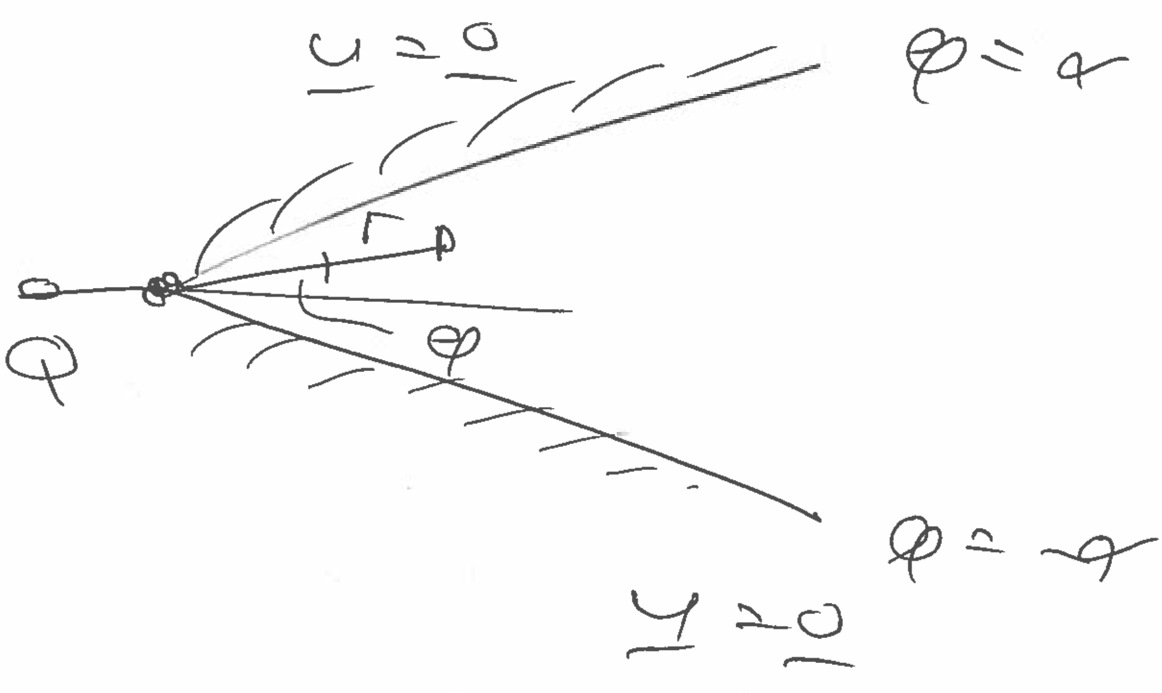


$$\underline{u} = \underline{u}(r, \varphi; \rho, \nu, Q, \alpha)$$

$$= \frac{\nu}{r} \underline{f}\left(\varphi; \alpha, \frac{Q}{\nu}\right)$$

$$\underline{u} = \frac{\nu}{r} f(\varphi)\, \underline{e_r}$$

---

r-mom. eqn:

$$u \frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} - \frac{u}{r^2}\right)$$

$$u = \frac{\nu}{r} f(\varphi) = \nu r^{-1} f(\varphi)$$

$$\frac{\partial u}{\partial r} = -\nu r^{-2} f(\varphi)$$

$$\frac{\partial^2 u}{\partial r^2} = 2\nu r^{-3} f(\varphi)$$

$$\frac{\partial^2 u}{\partial \varphi^2} = \nu r^{-1} f''(\varphi)$$

$v = 0 \qquad w = 0$

$$
\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial r} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} - \cancel{\frac{v^2}{r}} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\nabla^2 u - \frac{u}{r^2} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],
$$

$$
\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \nu\left[\cancel{\nabla^2 v} - \cancel{\frac{v}{r^2}} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right],
$$

$$
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \varphi} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w,
$$

$$
\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0.
$$

where

$$
\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.
$$

(2

$$\left(\frac{\nu}{r}f\right)\left(\frac{-\nu}{r^2}f'\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\cancel{-\nu\frac{1}{r^3}f} + \cancel{\frac{2\nu}{r^3}f} + \frac{\nu}{r^3}f'' - \cancel{\frac{\nu}{r^3}f}\right)$$

$$\boxed{-\frac{\nu^2}{r^3}f^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\nu^2}{r^3}f''} \quad (1)$$

φ-momentum:

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \nu\,\frac{2}{r^2}\frac{\partial u}{\partial \varphi}$$

$$\frac{\partial p}{\partial \varphi} = \rho\nu\frac{2}{r}\underbrace{\frac{\nu}{r}f'}_{\frac{\partial u}{\partial \varphi}} = \frac{2\nu^2\rho}{r^2}f'(\varphi)$$

$$\boxed{p(r,\varphi) = \frac{2\nu^2\rho}{r^2}f(\varphi) + g(r)} \quad (2)$$

into (1):

$$-\frac{\nu^2}{r^3}f^2 = -\frac{1}{\rho}\frac{\partial}{\partial r}\left(2\nu^2\rho r^{-2}f(\varphi) + g(r)\right) + \frac{\nu^2}{r^3}f''(\varphi)$$

$$-\frac{\nu^2}{r^3} f^2 = -\frac{1}{\rho}\left(-4\rho\nu^2 r^{-3} f(\theta) + \frac{dg(r)}{dr}\right) + \frac{\nu^2}{r^3} f''(\theta)$$

All terms go like r^{-3}

$$\frac{dg}{dr} = A r^{-3}$$ for some constant A.

$$g(r) = \underbrace{-\tfrac{1}{2}A}_{B}\, r^{-2} + C = \frac{B}{r^2} + C$$

set to zero since $f(r)$ only features in pressure

into (2)

$$p = \rho\left(\frac{2\nu^2 f}{r^2} + \frac{K}{r^2}\right) \qquad \rho K = B$$

into (1)

$$-\frac{\nu^2}{r^3} f^2 = -\frac{\partial}{\partial r}\left(\frac{2\nu^2 f + K}{r^2}\right) + \frac{\nu^2}{r^3} f''$$

$$-\frac{\nu^2}{\cancel{r^3}} f^2 = \frac{2(2\nu^2 f + K)}{\cancel{r^3}} + \frac{\nu^2}{\cancel{r^3}} f''$$

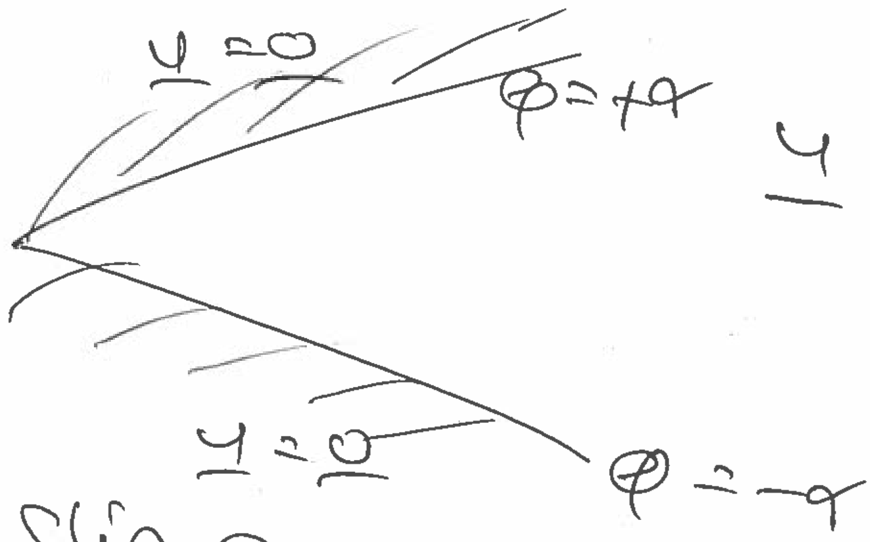
Differentiate eqn to get rid of the unknown constant K:

$$-\nu^2 2ff' = 4\nu^2 f' + \nu^2 f'''$$

$$\boxed{f''' + 4f' + 2ff' = 0}$$

3rd order nonlinear ODE for $f(\varphi)$

BC:



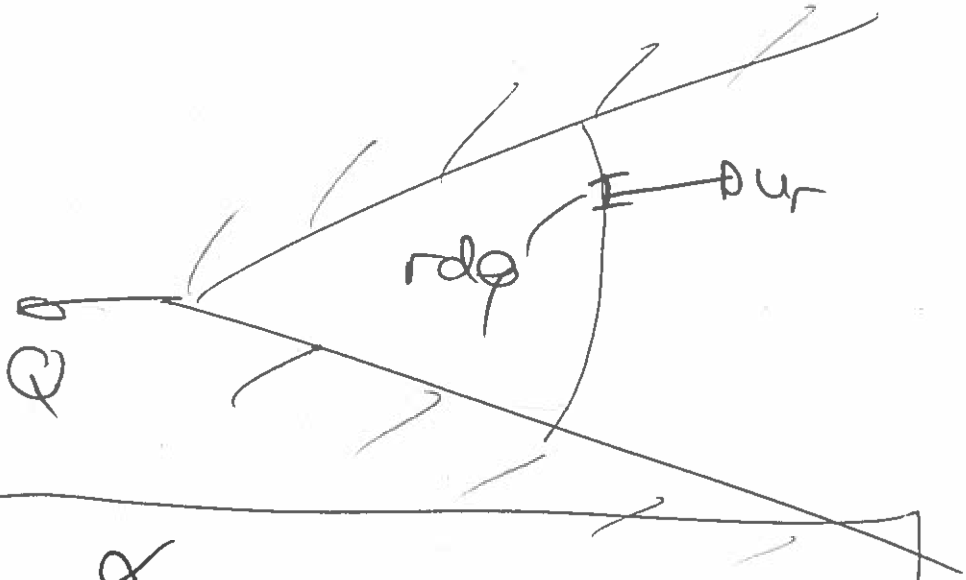
$\underline{u} = \frac{\nu}{r} f(\varphi)\, \underline{e_r}$

No slip @ $\varphi = \pm\alpha$:

$$f(\pm\alpha) = 0$$

Need to impose overall flow rate.

$$\int_{-\alpha}^{\alpha} u \, r \, d\varphi \overset{!}{=} -Q$$



$$\boxed{\int_{-\alpha}^{\alpha} r f(\varphi) \, d\varphi \overset{!}{=} -Q}$$

Eqns have to be solved numerically!

§ m-1 Streamfct & vorticity

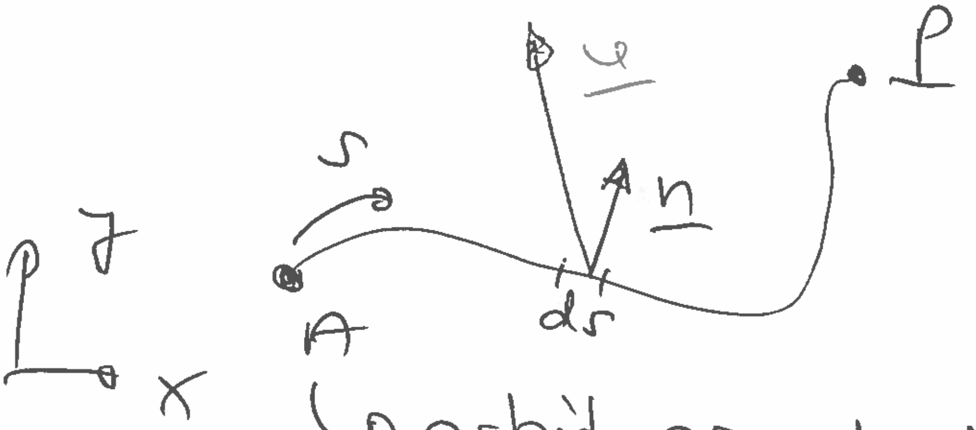
Alternative formulation of NSt; powerful in 2D & for incompressible fluids.

Streamfct (2D, incompr.)

$$\underline{u} = u\,\underline{e_x} + v\,\underline{e_y}$$

Def:

$$\psi_A(P) = \int_A^P \underline{u}\cdot\underline{n}\,ds$$



$\underline{n}$ to the left!!

↳ arbitrary but fixed

$\underline{u}\cdot\underline{n}$ = volume flux crossing the line.

$\psi_A(P)$ represents the volume flux (per unit depth in the z-direction) crossing the line AP.