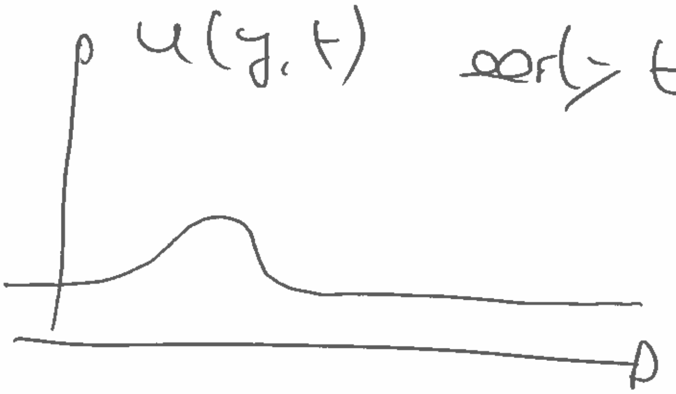
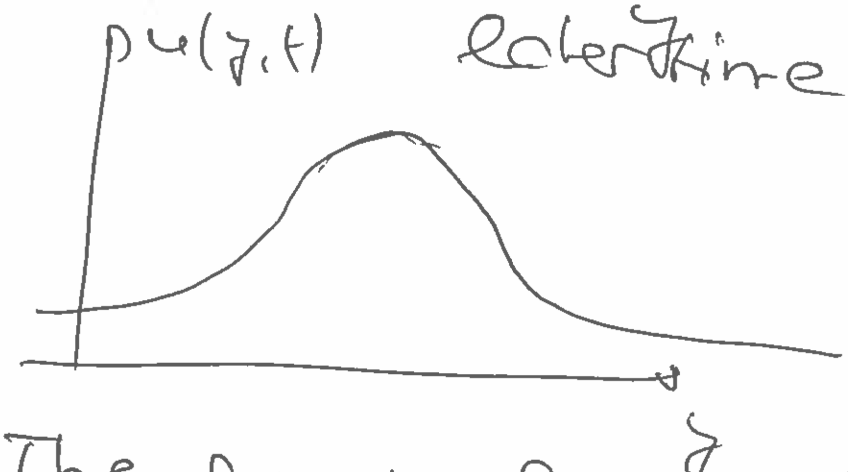
§7.2 Similarity solns

Often solutions to N.St. eqns. are self-similar, i.e. they have the same "shape" but possibly a different "scale" at different times.

E.g:





The generic form of such solutions is

$$u(y,t) = a(t)\, f\left(\frac{y}{b(t)}\right)$$

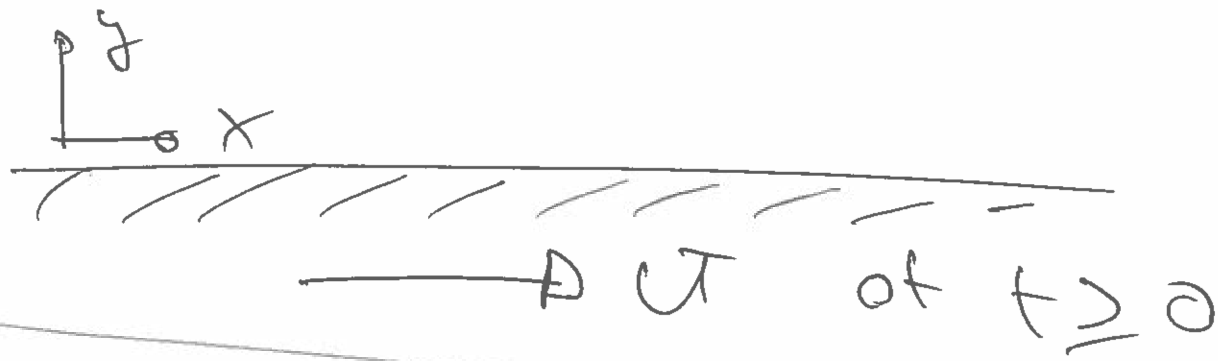
This scales the amplitude of the soln.

This scales the "width" of the solution.

- Similarity solns. don't always exist – try it!
- They often reduce PDEs to ODEs for $f(\xi)$ where $\xi = \frac{y}{b(t)}$ is the similarity variable.
- The existence of sim. solns. is often suggested by dimensionality arguments.

Example:

Rayleigh's jerked plate:



Parallel flow:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

BC: $u|_{y=0} = 0$ for $t = 0$

$u|_{y=0} = U$ for $t > 0$ (*)

$u \to 0$ as $y \to \infty$

IC: $u = 0$ for $t = 0$

$u(y, t; \nu, U)$

Note: • PDE is linear & homogeneous
• BC/ICs are homogeneous apart from (A)
• Have E&U

$\Rightarrow$ $u(y, t; \nu, U)$ has to be linear in U:

$$u(y, t; \nu, U) = U f(y, t; \nu)$$

Now: u & U have dimensions: m/sec

$\Rightarrow$ f has to be a dimensionless fct. of its arguments

$\Rightarrow$ f can only depend on a nondim. combination of its arguments!

Units:

$$[y] = m$$

$$[t] = sec$$

$$[\nu] = \frac{\left[\frac{\partial u}{\partial t}\right]}{\left[\frac{\partial^2 u}{\partial y^2}\right]} = \frac{\cancel{m}\ \cancel{sec}\ m^2}{sec^{\cancel{2}}\ \cancel{m}}$$

$$[\nu] = \frac{m^2}{sec}$$

So a non-dim. combination is:

$$\eta = \frac{y^2}{\nu t}\ ;\quad \eta = \frac{\nu t}{y^2}$$

$$\eta = \frac{\nu^2 t^2}{y^4}\ ;\quad \eta = \sin\left(\log\left(\frac{\nu t}{y^2}\right)\right)$$

etc.

Typically one tries to find a sim. variable that is linear in y.

$$u(y,t;\nu,U) = U f\left(\frac{y}{\sqrt{\nu t}}\right)$$

$$= U f\Big(\underbrace{y\,(\nu t)^{-1/2}}_{\eta}\Big)$$

Transform PDE, BC, IC:

$$\frac{\partial u}{\partial t} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial t}$$

$$\boxed{= U f' \frac{y}{\sqrt{\nu}}\left(-\frac{1}{2}\right) t^{-3/2}}$$

$$\frac{\partial u}{\partial y} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial y}$$

$$= U f' \cdot (\nu t)^{-1/2}$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = U f'' \frac{1}{\nu t}}$$

into PDE:

$$-\frac{1}{2}\cancel{U}\underbrace{\frac{y}{\sqrt{\nu t}}}_{\eta}\cancel{\frac{1}{t}} f' = \cancel{\nu}\cancel{U} f'' \frac{1}{\cancel{\nu}\cancel{t}}$$

$$\boxed{f'' + \frac{1}{2}\eta f' = 0}$$

ODE for $f(\eta)$

$\eta = \frac{y}{\sqrt{\nu t}}$

$u = U f(\eta)$

BC: $u = U$ at $y = 0$

$$\boxed{f(\eta = 0) = 1}$$

$u \to 0$ as $y \to \infty$

$$\boxed{f \to 0 \text{ as } \eta \to \infty}$$

IC: $u = 0$ at $t = 0$

$$\boxed{f \to 0 \text{ as } \eta \to \infty}$$

Same!

Here this works even though we have 3 BC/ICs for a 2nd order ODE!