$$\underline{u} = u_r \underline{e_r} + u_\varphi \underline{e_\varphi} + u_z \underline{e_z}$$

$$u_r = u$$
$$u_\varphi = v$$
$$u_z = w$$

To illustrate the origin of some of the terms in cyl. polars consider rigid body rotation:

$$\underline{u} = u_\varphi \underline{e_\varphi}$$

$$u_\varphi = \Omega r = v \qquad u, w = 0$$

radial mom. eqn:

$$+\frac{v^2}{r} = +\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\rho \frac{\Omega^2 r^{\cancel{2}}}{\cancel{r}} = \frac{\partial p}{\partial r}$$

$$\frac{\partial p}{\partial r} = \rho \Omega^2 r$$

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\cancel{\nabla^2 u} - \cancel{\frac{u}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \varphi} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right],$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \varphi} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w,$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0.$$

where

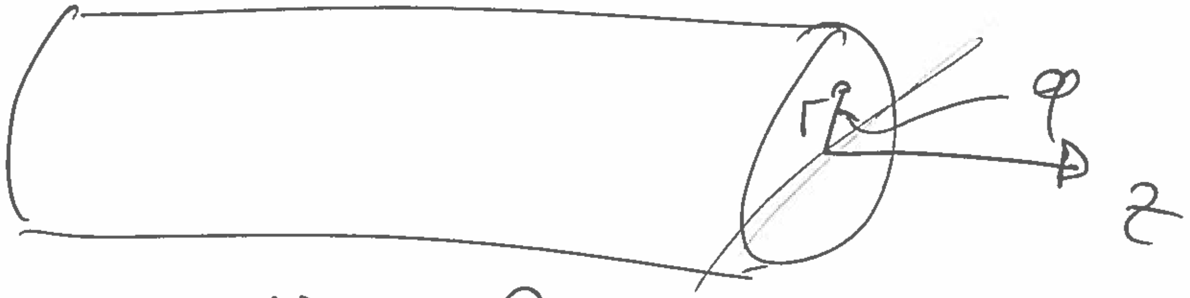
$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

$$p = \frac{1}{2}\rho\Omega^2 r^2 + p_0$$

p_0 const.

This is due to centrifugal "forces"

Example: Hagen-Poiseuille flow



pipe radius R.

Assume:

$$\underline{u} = u_z \underline{e_z} =$$

$$= \underbrace{u_z(r)}_{w(r)} \underline{e_z}$$

$$p = p(r, z)$$

$u = v = 0 \qquad w(r) \qquad \rho(\cancel{r}, z)$

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} - \cancel{\frac{v^2}{r}} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \cancel{\nu\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \varphi}\right]},$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = \cancel{-\frac{1}{\rho r}\frac{\partial P}{\partial \varphi}} + \cancel{\nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right]},$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial w}{\partial \varphi}} + \cancel{w\frac{\partial w}{\partial z}} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w,$$

$$\operatorname{div}\mathbf{u} = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(ru)} + \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

z - comp:

$w(r)$

$p(z)$

$$0 = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\text{fct of } z \text{ (at most)}} + \nu \underbrace{\overbrace{\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)}^{\nabla^2 w}}_{\text{fct of } r \text{ (at most)}}$$

As before

$$\frac{\partial p}{\partial z} = \text{const} = G$$

$$\frac{G}{\mu} = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)}_{\nabla^2 w}$$

$$\frac{G}{\mu} r = \frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$$

$$A + \frac{1}{2}\frac{G}{\mu} r^2 = r\frac{\partial w}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{1}{2}\frac{G}{\mu} r + \frac{A}{r}$$

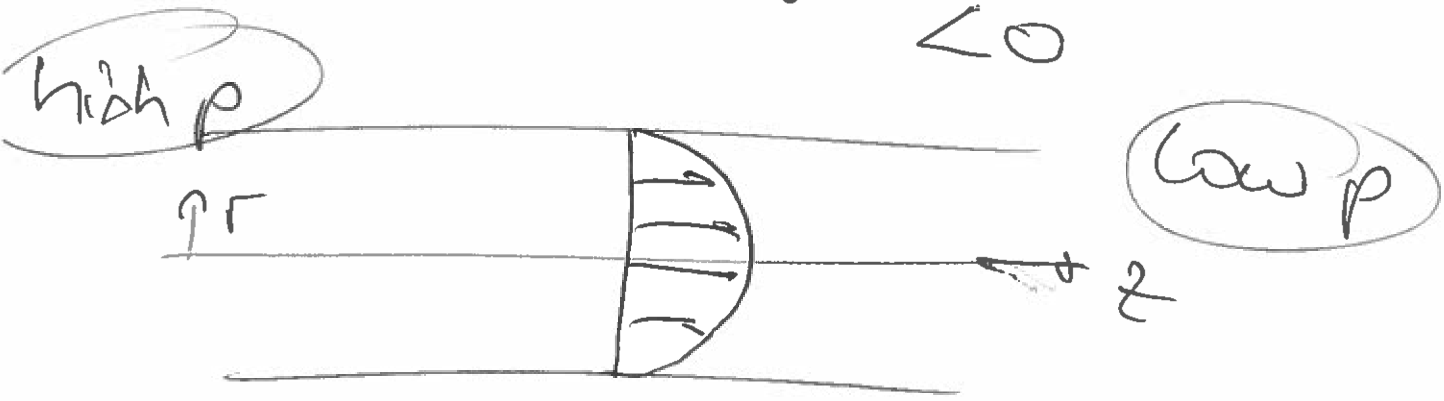
$$w(r) = \frac{1}{4}\frac{G}{\mu}r^2 + A\ln r + B$$

A, B from BCs:

$w(r=R) = 0$ (no slip)

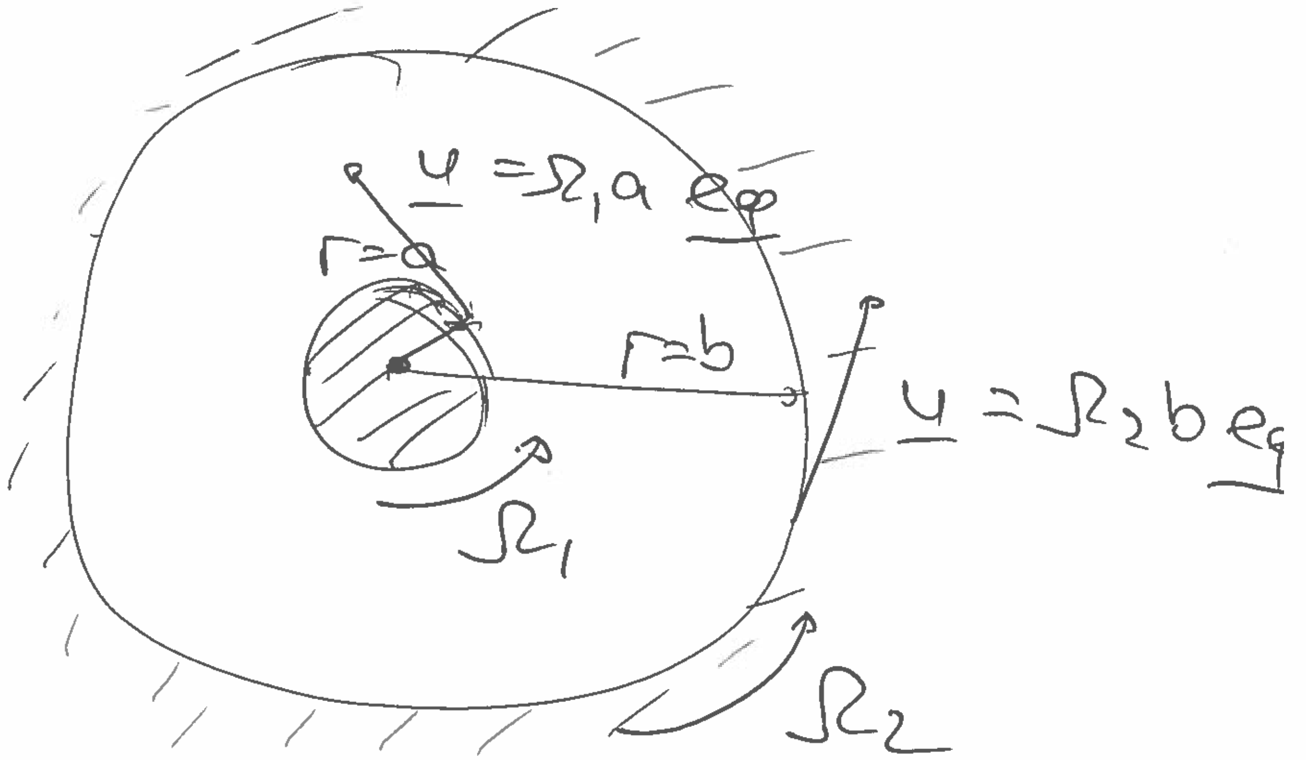
$w(r=0)$ finite : $A = 0$

$$w(r) = \frac{1}{4}\frac{G}{\mu}\underbrace{(r^2 - R^2)}_{<0}$$



$G = \frac{\partial p}{\partial z} < 0$ for sketch to be correct ✓

Example: Circular Couette flow



Assumptions

$$\left.\begin{array}{l} \cdot \text{ steady} \\ \cdot\ \underline{u} = u_\varphi \underline{e_\varphi} \\ \cdot\ \dfrac{\partial}{\partial \varphi} = 0 \\ \cdot\ \dfrac{\partial}{\partial z} = 0 \end{array}\right\} \underline{u} = u_\varphi(r)\, \underline{e_\varphi}$$

Flow driven by wall (Couette)

$$\nabla p = \underline{0}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \varphi}+w\frac{\partial u}{\partial z}-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}+\nu\left[\nabla^2 u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial \varphi}\right],$$

$$\cancel{\frac{\partial v}{\partial t}}+\cancel{u\frac{\partial v}{\partial r}}+\cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}}+\cancel{w\frac{\partial v}{\partial z}}+\cancel{\frac{uv}{r}} = \cancel{-\frac{1}{\rho r}\frac{\partial P}{\partial \varphi}}+\nu\left[\nabla^2 v-\frac{v}{r^2}+\cancel{\frac{2}{r^2}\frac{\partial u}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial w}{\partial t}}+\cancel{u\frac{\partial w}{\partial r}}+\cancel{\frac{v}{r}\frac{\partial w}{\partial \varphi}}+\cancel{w\frac{\partial w}{\partial z}} = \cancel{-\frac{1}{\rho}\frac{\partial P}{\partial z}}+\cancel{\nu\nabla^2 w},$$

$$\operatorname{div}\mathbf{u} = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(ru)}+\cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}}+\cancel{\frac{\partial w}{\partial z}} = 0.$$

where $u = u_r = 0$ $\quad u_\varphi = U(r)$

$w = u_z = 0$ $\quad \nabla \rho = 0$

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}+\frac{\partial^2}{\partial z^2}.$$

φ - mom. eqn:

$$0 = \nu \left(\underbrace{\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}}_{D^2 v} - \frac{v}{r^2} \right)$$

$$r^2 \frac{d^2 v}{dr^2} + r \frac{dv}{dr} - v = 0$$

Euler ODE

$$v \sim r^\lambda$$