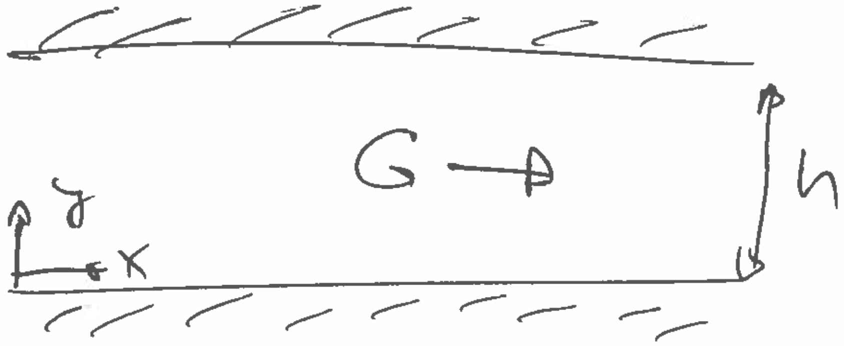
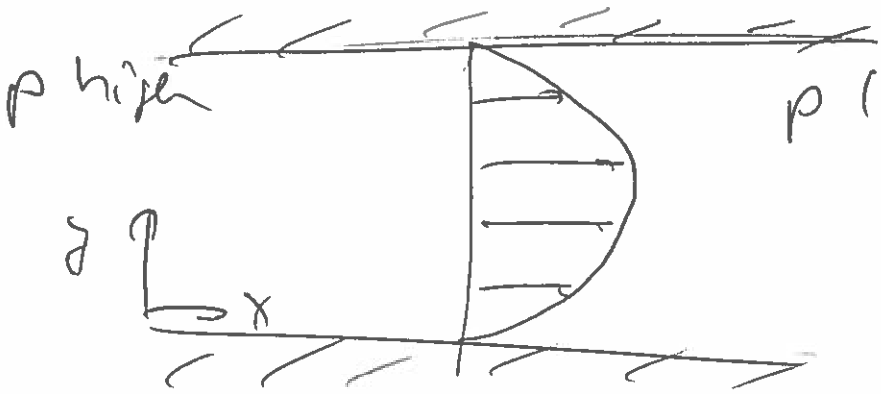
Poiseuille flow



$$u(y,z,t) = u(y)$$

$$G = \mu \frac{d^2u}{dy^2} \qquad u(0) = u(h) = 0$$

$$\Rightarrow \boxed{u = \frac{G}{2\mu}\underbrace{(y^2 - hy)}_{<0}}$$

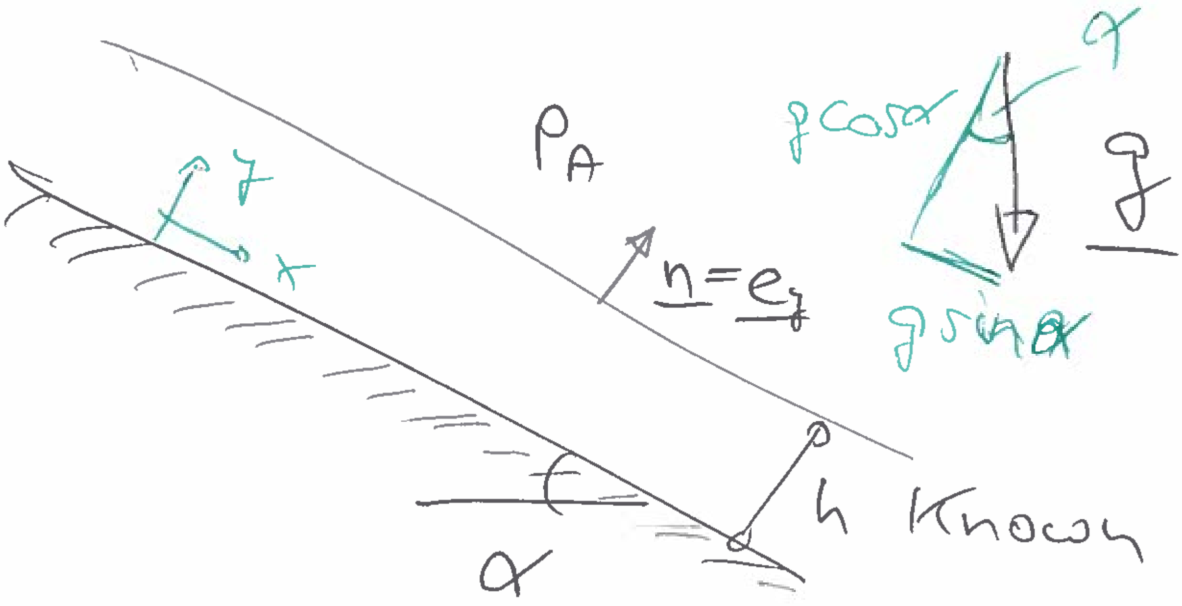


$u \geq 0$

$\Rightarrow G < 0$

$$G = \frac{dp}{dx} < 0$$

Example: flow down on inclined plane



$$\underline{F} = \underline{g} = \underbrace{g\sin\alpha}_{f_x}\,\underline{e}_x \underbrace{-g\cos\alpha}_{f_y}\,\underline{e}_y$$

Assume: parallel flow, steady & indep. of z.

$$\cancel{\rho\frac{\partial u}{\partial t}} = \underbrace{\rho g\sin\alpha}_{f_x} - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\boxed{0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + \rho g\sin\alpha} \quad (1)$$

y - comp:

$$\boxed{0 = -\frac{\partial p}{\partial y} \underbrace{- \rho g \cos\alpha}_{f_y}} \quad (2)$$

z - comp:

$$\boxed{0 = -\frac{\partial p}{\partial z} + \underbrace{0}_{f_z}} \Rightarrow p = p(x,y)$$

BC: No slip @ $y = 0$:

$$\boxed{u(y=0) = 0}$$

At $y = h$: Air is far less viscous than the fluid.
$\Rightarrow$ Traction acting onto the fluid is given by the air pressure p_A.

$$\underline{t} = \underbrace{-p_A \underline{n}}_{\text{applied traction}} \qquad \underline{n} = \underline{e_y}$$

$$\underline{t} = t_x \underline{e_x} + t_y e_y + t_z \underline{e_z}$$

$$\underline{t} = t_1 \underline{e_1} + t_2 \underline{e_2} + t_3 \underline{e_3}$$

$$t_1 = 0 , \quad t_2 = -p_A , \quad t_3 = 0$$

$$n_1 = 0 , \quad n_2 = 1 , \quad n_3 = 0$$

$$t_i = \tau_{ij} n_j$$

$$t_i = \left[-p \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] n_j$$

at $y = x_2 = h$

$$t_i = -p n_i + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_j$$

note: only n_2 is nonzero

for $i = 1, 2, 3$

<u>$i = 2$:</u>

because $u_2 = v = 0$

$$t_2 = -p_A = -p \overset{1}{n_2} + \mu \left(\cancel{\frac{\partial u_2}{\partial x_2}} + \cancel{\frac{\partial u_2}{\partial x_2}} \right) n_2$$

(other terms in sum vanish because $n_1 = n_3 = 0$)

$$\boxed{p(y=h) = p_A}$$

<u>i=1:</u>

$$t_1 = 0 = -p\cancel{n_1} + \mu\left(\frac{\partial u_1}{\partial x_2} + \cancel{\frac{\partial u_2}{\partial x_1}}\right) n_2$$

$$\boxed{0 = \mu \frac{\partial u}{\partial y} \quad \text{at} \quad y = h}$$

(no tangential shear stress)

<u>i=3:</u> $0 = 0$ (EXERCISE)

Integrate (2) w.r.t. y

$$\frac{\partial p}{\partial y} = -\rho g \cos\alpha$$

$$p(x,y) = -\rho g \cos\alpha \, y + f(x)$$

Apply pressure BC:

$$p_A = p(y=h) = -\rho g \cos\alpha \, h + f(x)$$

$$\boxed{p(x,y) = p_A + \rho g \cos\alpha \, (h-y)}$$

(actually indep. of x)

Note: pressure increases through thickness of fluid layer ✓

Now (1):

$$0 = -\cancel{\frac{\partial p}{\partial x}} + \underbrace{\rho g \sin\alpha}_{\text{acts like press gradient}} + \mu \frac{\partial^2 u}{\partial y^2}$$

Integrate twice:

$$u(y) = -\frac{1}{2}\frac{\rho g \sin\alpha}{\mu} y^2 + Ay + B$$

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BC: $u(y=0) = 0$

$$\mu \frac{\partial u}{\partial y} = 0 \quad \text{at } y = h$$

$$u(y) = \frac{g \sin\alpha}{\nu}\left(hy - \frac{1}{2}y^2\right)$$