§1 Index notation

Various ways of expressing a vector:

$$\underline{a} = \underbrace{a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3}_{\text{in components relative to basis}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

symbolic

in components relative to basis

in component

$$(\underline{e}_1, \underline{e}_2, \underline{e}_3) = (\underline{i}, \underline{j}, \underline{k})$$

Convention 1:

Simply write down one generic term of vector/vector-eqn.

$$\underline{c} = \underline{a} + \underline{b} \qquad c_i = a_i + b_i$$

where i is a free index which takes values $i = 1, 2, 3$

Example:

$$\nabla \phi = \frac{\partial \phi}{\partial x_1}\underline{e}_1 + \frac{\partial \phi}{\partial x_2}\underline{e}_2 + \frac{\partial \phi}{\partial x_3}\underline{e}_3 =$$

$$= \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix} \Rightarrow \frac{\partial \phi}{\partial x_i}$$

Convention 2: Summation convention:

Rule: Sum automatically over repeated indices (dummy indices)

Example:

$$\alpha = \underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$= \cancel{\sum_{i=1}^{3}} u_i v_i = \underbrace{u_i v_i}_{\text{Summation implied}}$$

Note: summation index does not appear in result!

$\underline{u} \cdot \underline{v} = u_i v_i = u_k v_k$ etc.

Summation index = dummy index.

Example:

$$\operatorname{div} \underline{u} = \operatorname{div} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} =$$

$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= \frac{\partial u_i}{\partial x_i} = \frac{\partial u_j}{\partial x_j}$$

Higher order "tensors"

So far: index notation to represent vectors (3 components represented by one free index).

Higher order tensors arise naturally in many applications.

Example:

$$\underline{\sigma} = \underset{\sim}{T} \cdot \underline{n}$$

stress vector — stress tensor (matrix) — normal vector

$$\sigma_i = T_{i\underbrace{j} n_j}_{\text{sum!}}$$

matrix vector product!

Note: Every term in a vector eqn. has to have the same free index!

One special second-order tensor is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$[\delta_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(unit matrix)

δ_{ij} has an interesting property when used in summation:

$$b_j = \underbrace{a_i \, \delta_{ij}}_{} = \sum_{i=1}^{3} a_i \delta_{ij} = a_j$$

"δ_{ij} exchanges indices".