

Chapter 8

The Streamfunction and Vorticity

- For 2D incompressible flows, it is possible to recast the Navier-Stokes equations in an alternative form in terms of the streamfunction and the vorticity.
- In many applications, the streamfunction-vorticity form of the Navier-Stokes equations provides better insight into the physical mechanisms driving the flow than the ‘primitive variable’ formulation in terms of u, v and p .
- The streamfunction and vorticity formulation is also useful for numerical work since it avoids some problems resulting from the discretisation of the continuity equation.

Unless specifically stated, all results in this chapter are restricted to 2D incompressible flows.

8.1 The Streamfunction

- The streamfunction is defined as

$$\psi_A(P) = \int_A^P \mathbf{u} \cdot \mathbf{n} \, ds, \quad (8.1)$$

where the integral has to be evaluated along a line from the arbitrary but fixed point A to point P. \mathbf{n} is the unit normal on the line from A to P. We regard $\psi_A(P)$ as a function of the location of point P.

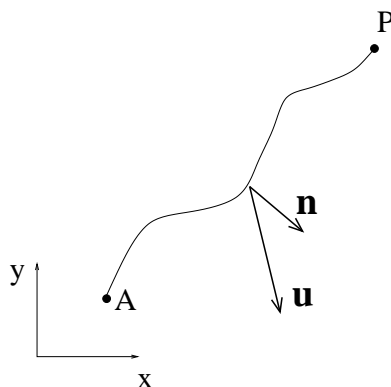


Figure 8.1: Sketch illustrating the definition of the streamfunction.

- The sketch in Fig. 8.1 shows that $\mathbf{u} \cdot \mathbf{n}$ is equal to the component of the velocity \mathbf{u} that crosses the line AP. Therefore $\psi_A(P)$ represents the volume flux (per unit depth in the z -direction) through the line between A and P.

- Evaluating $\psi_A(P)$ along two different paths and invoking the integral form of the incompressibility constraint shows that $\psi_A(P)$ is path-independent, i.e. its value only depends on the locations of the points A and P.
- Changing the position of point A only changes $\psi_A(P)$ by a constant. It turns out that for all applications such changes are irrelevant. It is therefore common to suppress the explicit reference to A. Hence, we regard $\psi_A(P)$ as a function of the spatial coordinates only, i.e. $\psi_A(P) = \psi(P) = \psi(x, y)$.
- Streamlines are lines which are everywhere tangential to the velocity field, i.e. $\mathbf{u} \cdot \mathbf{n} = 0$, where \mathbf{n} is the unit normal to the streamline. Hence the streamfunction ψ is constant along streamlines.
- Note that stationary impermeable boundaries are also characterised by $\mathbf{u} \cdot \mathbf{n} = 0$, where \mathbf{n} is the unit normal on the boundary. Therefore, ψ is also constant along such boundaries.
- Invoking the integral incompressibility constraint for an infinitesimally small triangle shows that ψ is related to the two cartesian velocity components u and v via

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{8.2}$$

- Similarly, in plane cylindrical polars, the velocity components are given by

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \varphi} \quad \text{and} \quad u_\varphi = -\frac{\partial \psi}{\partial r}. \tag{8.3}$$

- Flows which are specified by a streamfunction automatically satisfy the continuity equation since

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial \psi}{\partial x} \right) = 0. \tag{8.4}$$

- For 2D flows, the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ only has one non-zero component (in the z -direction), i.e. $\boldsymbol{\omega} = \omega \mathbf{e}_z$ where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{8.5}$$

Using the definition of the velocities in terms of the streamfunction shows that

$$\omega = \frac{\partial}{\partial x} \left(-\frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} \right) \tag{8.6}$$

and therefore

$$\omega = -\nabla^2 \psi, \tag{8.7}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the 2D Laplace operator.

8.2 The Streamfunction-Vorticity form of the Navier-Stokes equations

- Straightforward algebraic manipulation of the 3D momentum equations transforms them into the *vorticity transport equation*

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega} \tag{8.8}$$

(see the separate handout for the derivation; this equation is valid in 3D).

- This equation shows that the rate of change of the vorticity of material particles, $D\boldsymbol{\omega}/Dt$, is controlled by ‘vortex stretching’ (described by $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$; this is a familiar result from inviscid fluid mechanics) and by diffusion (described by $\nu \nabla^2 \boldsymbol{\omega}$). The diffusion of vorticity only occurs in viscous flows.

- For 2D flows, vortex stretching is absent since $\mathbf{u} = u(x, y) \mathbf{e}_x + v(x, y) \mathbf{e}_y$ and $\boldsymbol{\omega} = \omega(x, y) \mathbf{e}_z$ and therefore $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = 0$.
- For 2D flows, the scalar vorticity transport equation

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega \quad (8.9)$$

together with the equation for the vorticity in terms of the streamfunction

$$\omega = -\nabla^2 \psi \quad (8.10)$$

and

$$u = \partial\psi/\partial y \quad \text{and} \quad v = -\partial\psi/\partial x \quad (8.11)$$

provide the streamfunction-vorticity formulation of the Navier-Stokes equations. It consists of only two PDEs for the scalars ω and ψ rather than the three PDEs for u, v and p in the ‘primitive variable’ form.

- In the limit of zero Reynolds number, only one fourth-order PDE for the streamfunction ψ needs to be solved, namely the biharmonic equation

$$\nabla^4 \psi = 0, \quad (8.12)$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \quad (8.13)$$

This can be shown by, e.g., taking the curl of the Stokes equations.

8.2.1 The streamfunction and the biharmonic equation in cylindrical polars

- In cylindrical polars, (r, φ) the relation between the streamfunction $\psi(r, \varphi)$ and the velocity components u_r and u_φ is:

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \varphi} \quad (8.14)$$

and

$$u_\varphi = -\frac{\partial \psi}{\partial r}, \quad (8.15)$$

where $\mathbf{u} = u_r \mathbf{e}_r + u_\varphi \mathbf{e}_\varphi$.

- The biharmonic equation in polar coordinates:

$$\nabla^4 \psi(r, \varphi) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right] \left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} \right] \quad (8.16)$$

$$\nabla^4 \psi(r, \varphi) = \psi_{,rrrr} + \frac{2}{r} \psi_{,rrr} - \frac{1}{r^2} (\psi_{,rr} - 2\psi_{,rr\varphi\varphi}) + \frac{1}{r^3} (\psi_{,r} - 2\psi_{,r\varphi\varphi}) + \frac{1}{r^4} (4\psi_{,\varphi\varphi} + 2\psi_{,\varphi\varphi\varphi\varphi}) \quad (8.17)$$

- For axisymmetric solutions:

$$\nabla^4 \psi(r) = \frac{1}{r} \left[r \left(\frac{1}{r} [r\psi_{,r}]_{,r} \right)_{,r} \right]_{,r} \quad (8.18)$$

$$\nabla^4 \psi(r) = \psi_{,rrrr} + \frac{2}{r} \psi_{,rrr} - \frac{1}{r^2} \psi_{,rr} + \frac{1}{r^3} \psi_{,r} \quad (8.19)$$

- The general form of the solution of the biharmonic equation in cylindrical polars is known. It can be represented by superposition of the following solutions:

– The general axisymmetric solution:

$$\psi(r) = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r \quad (8.20)$$

– The general separated non-axisymmetric solution:

For $n = 1$:

$$\begin{aligned} \psi(r, \varphi) = & \left(Ar + \frac{B}{r} + Cr^3 + Dr \ln r \right) \cos(\varphi) \\ & + \left(ar + \frac{b}{r} + cr^3 + dr \ln r \right) \sin(\varphi) \end{aligned} \quad (8.21)$$

For $n \geq 2$:

$$\begin{aligned} \psi(r, \varphi) = & \sum_{n=2}^{\infty} (A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}) \cos(n\varphi) \\ & + (a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{-n+2}) \sin(n\varphi) \end{aligned} \quad (8.22)$$

The coefficients $(A_0, B_0, C_0, D_0, A_1, B_1, C_1, D_1, a_1, b_1, c_1, d_1, A_2, B_2, C_2, D_2, a_2, b_2, c_2, d_2, \dots)$ have to be determined from the boundary conditions.