HIGH REYNOLDS NUMBER FLOWS & BOUNDARY LAYERS

- Recall the derivation of the scaled Navier-Stokes equations (using an inertial pressure scale $p = \rho U^2\, \tilde{p}$):

$$\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\tilde{\nabla}^2 \tilde{u}_i$$

- The Reynolds number

$$Re = \frac{Ua}{\nu}$$

  of the flow was formed with

  - the typical velocity scale for the flow U,
  - the typical length scale a over which the velocity undergoes characteristic changes and
  - the kinematic viscosity ν.

- Considering the limit $Re \to \infty$ (which corresponds to high speed flows over large length scales and small viscosity), shows that, as a first approximation, such flows can be described by the *Euler equations*

$$\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i},$$

  which are the equations of inviscid fluid flow.

- Note that the order of the Euler equations is lower than that of the full Navier Stokes equations (first rather than second spatial derivatives!). This means that not all boundary conditions can be applied on the surface of a solid body.

- Typically, the no-slip condition is discarded in favour of the no-penetration condition (see inviscid flow theory).

- However, close to the surface of the body, the no-slip condition always becomes important since viscosity (no matter how small) will always reduce the fluid velocity to zero as one approaches the surface of the solid body. This manifests itself in the existence of a thin layer (a so-called boundary layer) in which viscous effects are important and in which the velocity varies rapidly to fulfill the no-slip condition:

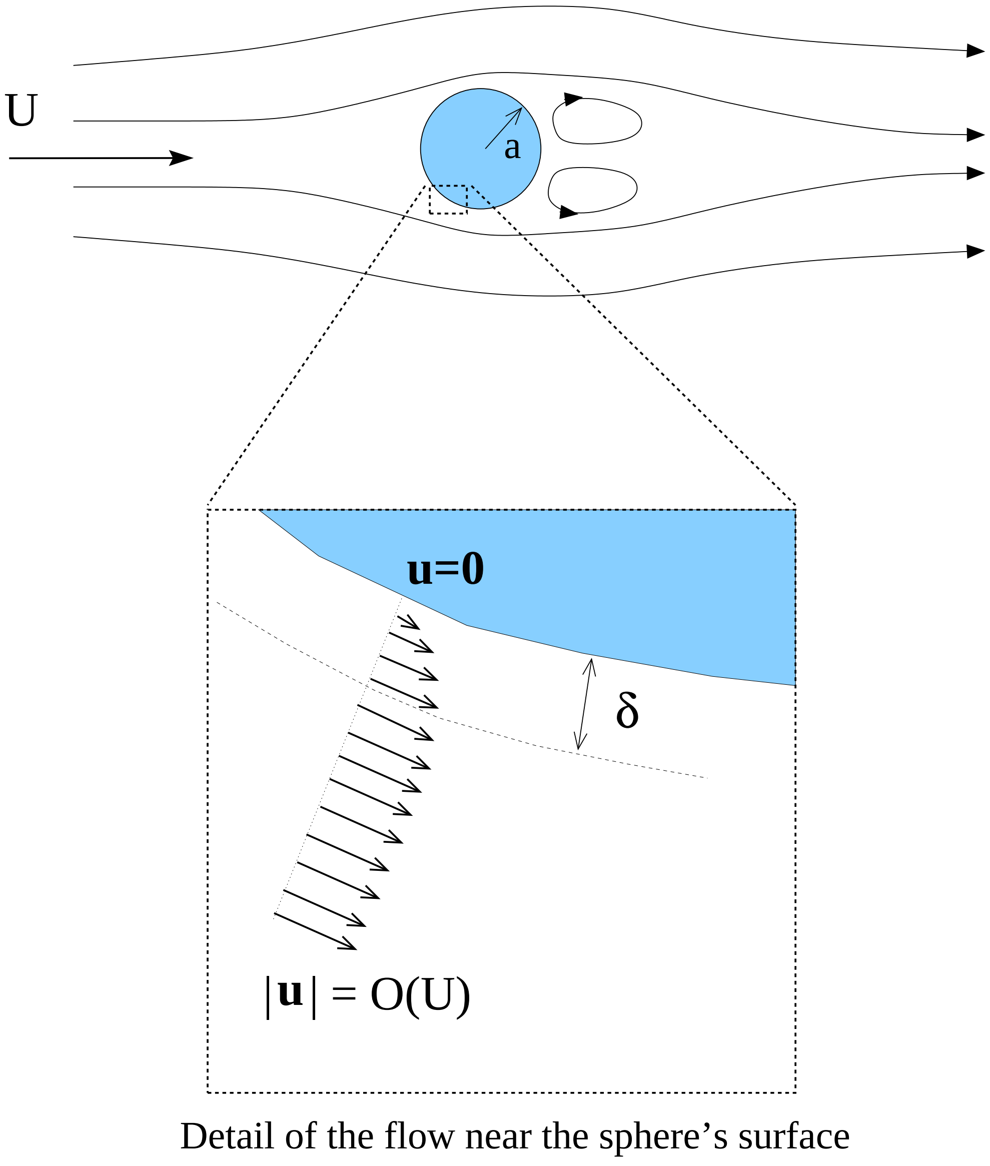

Figure 1: Boundary layer: Over length scales comparable with the size of the sphere, a, the flow behaves like an inviscid flow since $Ua/\nu \gg 1$ (see upper half of the sketch). Close to the surface of the sphere, viscosity enforces the no-slip condition. This leads to rapid variations of the velocity (whose size is still $|\mathbf{u}| = O(U)$) over short distances $\delta \ll a$.

Illustrative example for the occurrence of boundary layers

- Mathematically, the limit $Re \to \infty$ represents a *singular limit* and appropriate simplified solutions of the Navier-Stokes equations have to be found by matched asymptotic expansions:
- The inviscid solution which is valid at sufficiently large distances from the surface of the solid body represents the 'outer solution' which has to be matched to an 'inner solution' which represents the flow in the boundary layer.
- To illustrate the mathematical mechanism behind the formation of a boundary layer, consider the model equation:

$$\epsilon u'' + u' = 0, \tag{1}$$

subject to a 'no-slip' condition on the 'surface',

$$u(0) = 0,$$

and the asymptotic approach to the 'free stream' velocity U far away from the 'surface',

$$u \to U \quad \text{as} \quad x \to \infty.$$

- Fig. 2 shows the exact solution

$$u(x) = U\,(1 - \exp(-x/\epsilon))$$

for various values of the 'small' parameter ϵ.

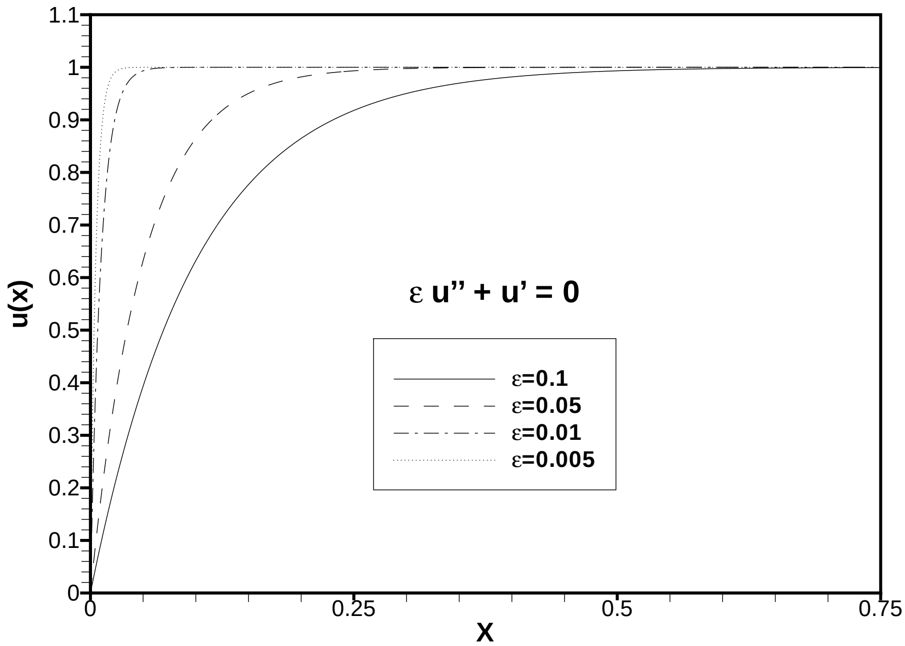

Figure 2: A model boundary layer for $U = 1$

- If we set $\epsilon = 0$, in (1) we obtain a first order equation which we can only subject to one boundary condition (e.g. $u \to U$ as $x \to \infty$) and the solution is $u_{\epsilon\equiv 0}(x) \equiv U$.

- This is different from actually performing the limit $\epsilon \to 0$: The smaller ϵ, the better $u(x)$ is approximated by $u_{\epsilon\equiv 0}(x) = U$. However, for *any* finite ϵ (no matter how small), the solution undergoes a rapid variation inside a narrow 'boundary layer' near $x = 0$ to also fulfill the second boundary condition $u(0) = 0$.

The boundary layer equations

- We will now derive the equations which govern the flow in the boundary layer.
- They will turn out to be simpler than the full Navier-Stokes equations (that's why they are useful!) but more complicated than the Euler equations which describe the flow at a sufficiently large distance from the surface.
- Remember that we are interested in the flow in a very thin layer close to the surface of the solid body. Since the layer is very thin, we can neglect the surface curvature (as a first approximation) and introduce a local cartesian coordinate system whose x-axis is aligned with the surface.
- Boundary layer scales (for 2D steady flow):

  x-scale: a

  y-scale: $\delta \ll a$ (but as yet unknown)

  u-scale: U

  v-scale: V (as yet unknown)

  p-scale: $P = \rho U^2$ (the inertial pressure scale is appropriate).
- Note that in the main flow, we non-dimensionalised all lengths by the global length scale a, whereas in the boundary layer, we use different scales for the x and y-directions. To distinguish the two sets of non-dimensional variables, we will use a hat to identify variables which have been non-dimensionalised by the boundary layer scales, e.g. $y = \delta \hat{y} = a \tilde{y}$.

- As in previous examples of scaling, we must ensure that the terms in the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

are balanced. This provides the v-scale (as in lubrication theory) as

$$V = \left(\frac{\delta}{a}\right) U \ll U.$$

- With this scaling, the x-momentum equation becomes

$$\frac{\rho U^2}{a}\left(\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}}\right) = -\frac{\rho U^2}{a}\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\mu U}{\delta^2}\left(\underbrace{\left(\frac{\delta}{a}\right)^2 \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}}_{\ll 1} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}\right).$$

- We neglect the first term in the scaled Laplace operator and obtain

$$\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{\partial \hat{p}}{\partial \hat{x}} + \underbrace{\left(\frac{\mu a}{\rho U \delta^2}\right)}_{=O(1)}\frac{\partial^2 \hat{u}}{\partial \hat{y}^2}$$

- Now we use our knowledge about the balance of forces in the boundary layer: The inertial forces (which dominate outside the boundary layer) have to be balanced by viscous forces (which reduce the velocity to zero on the surface). Hence the viscous and inertial terms have to be of the same order of magnitude. This can only be the case if the term in the round brackets is of order $O(1)$.

- This provides a scale for the boundary layer thickness

$$\delta = \sqrt{\frac{a\mu}{\rho U}} = \sqrt{\frac{\nu a^2}{Ua}} = \frac{a}{\sqrt{Re}},$$

and yields the scaled and simplified version of the x-momentum equation

$$\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}.$$

- Note that the boundary layer thickness decreases as $\delta \sim Re^{-1/2}$ as $Re \to \infty$.

- The same scaling applied to the y-component of the momentum equations shows that to leading order

$$\frac{\partial \hat{p}}{\partial \hat{y}} = 0,$$

which is again similar to lubrication theory.

The boundary layer equations and the associated boundary conditions

- To summarise, the flow in the boundary layer is governed by

$$\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2},$$

$$\frac{\partial \hat{p}}{\partial \hat{y}} = 0,$$

and

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0.$$

- The boundary layer equations present a parabolic system of PDEs (note that the highest derivative in the x-direction is of first order).

- The boundary and initial conditions are derived by matching the flow in the boundary layer to the inviscid Euler flow $\mathbf{u}_E$ 'outside' the boundary layer.

- For the matching process, remember that the y-coordinate in the Euler flow ($y = a\tilde{y}$) is scaled differently from the y-coordinate in the boundary layer ($y = \delta\hat{y}$).

- Since $a \gg \delta$, the matching (which must be performed at the 'outer edge' of the boundary layer) can be carried out at $\hat{y} \to \infty$ and $\tilde{y} = 0$, in the respective coordinates.

- The physical basis for this approximation is the assumption that the presence of the very thin boundary layer does not affect the outer inviscid flow. Similarly, since the changes in the boundary layer take place over very short distances, we see that on the length scale of the boundary layer, the velocity distribution in the outer inviscid flow near the solid surface appears to be constant (and is thus given by the value at $\tilde{y} = 0$).

- Hence, at the present level of approximation, the solution of the boundary layer problem consists of two steps:

  1. Solve the outer inviscid flow problem, subject to the no-penetration condition to determine the inviscid Euler flow field $\mathbf{u}_E(\tilde{x}, \tilde{y})$. This flow field has a non-zero slip velocity on the surface, i.e. $\mathbf{u}_E(\tilde{x}, \tilde{y} = 0) \neq \mathbf{0}$.
  2. Solve the corresponding boundary layer problem and match the boundary layer velocity 'far from the surface' to the slip velocity predicted by the inviscid flow.

- The boundary and initial conditions are:

  – At the 'upstream edge' of the boundary layer, the velocity distribution has to be prescribed (e.g. the undisturbed inviscid velocity distribution at the leading edge of a flat plate, as in Fig. 3)

  – On the surface of the body, the boundary layer equations allow us to fulfill the no-slip condition.

  – 'Far away' from the surface of the solid body (i.e. as $\hat{y} \to \infty$), the velocity distribution in the boundary layer has to approach the inviscid Euler velocity distribution at the surface of the body, i.e.

$$\mathbf{u}(x, \hat{y} \to \infty) = \mathbf{u}_E(x, \tilde{y} = 0).$$

  – The pressure is constant throughout the thickness of the boundary layer and is therefore determined by the pressure distribution in the inviscid Euler flow, i.e.

$$p(x, \hat{y}) = p(x) = p_E(x, \tilde{y} = 0).$$

- Fig. 3 illustrates the required boundary and initial conditions for the case of the flow past a thin plate which is aligned with the direction of the flow.

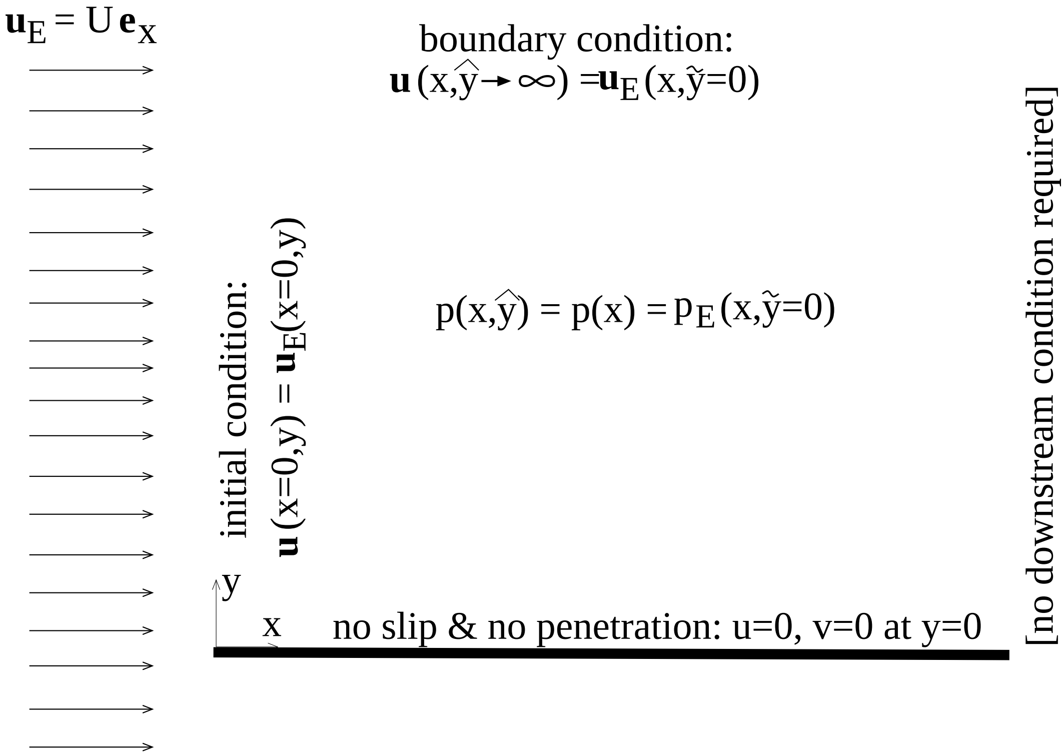

Figure 3: Boundary and initial conditions for the boundary layer equations, illustrated for the case of the flow past a thin flat plate, aligned with the uniform far-field flow, $\mathbf{u}_E = U\mathbf{e}_x$. Note the different non-dimensional y-coordinates inside the boundary layer and in the Euler region.

Remarks:

- Note that the boundary layer equations are non-linear. Analytical solutions are only known for a few special cases and in general numerical solution techniques have to be employed.
- The continuity equation and the transverse velocity v can be eliminated by formulating the problem in terms of a streamfunction $\psi(x, y)$.
- Throughout the derivation we have assumed that viscous effects remain confined to the boundary layer. This assumption breaks down when the boundary layer separates from the surface.
- Boundary layer separation occurs frequently in the flow past blunt bodies (see, e.g., the recirculation area behind the sphere in Fig. 1).
- In aircraft design, one tries to design the wing profile such that the boundary layer remains attached to the wing in all operating conditions. Therefore, boundary layer theory is an extremely powerful tool in aerodynamics.