$$\boxed{\begin{aligned} &\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} \\ &\frac{\partial u_j}{\partial x_j} = 0 \end{aligned}}$$ (1

+ BC & IC

N. St. eqn. are very complicated.
We cannot find general solns to these eqns because they are nonlinear, but can find solutions in certain special cases.

§ 5 (?) Parallel flows

Assume that the flow is unidirectional.
w.l.o.g.: choose coordinates such that

$$\underline{u} = u(x, y, z, t)\,\underline{e_x}$$

<u>Note:</u>

$x_1 = x \qquad u_1 = u$

$x_2 = y \qquad u_2 = v$

$x_3 = z \qquad u_3 = w$

3

$u = u(\cancel{x}, y, z, t) \qquad v = 0 \qquad w = 0$

$u = u(y, z, t)$

$$\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v}\frac{\partial u}{\partial y} + \cancel{w}\frac{\partial u}{\partial z} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + \cancel{w\frac{\partial v}{\partial z}} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right)$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial x}} + \cancel{v\frac{\partial w}{\partial y}} + \cancel{w\frac{\partial w}{\partial z}} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\cancel{\frac{\partial^2 w}{\partial x^2}} + \cancel{\frac{\partial^2 w}{\partial y^2}} + \cancel{\frac{\partial^2 w}{\partial z^2}}\right)$$

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

$\nu = \frac{\mu}{\rho}$

Kinematic viscosity

Remaining eqns:

$$\boxed{\rho \frac{\partial u}{\partial t} = \rho f_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)} \quad (1)$$

$$\boxed{\rho f_y = \frac{\partial p}{\partial y}} \quad (2)$$

$$\boxed{\rho f_z = \frac{\partial p}{\partial z}} \quad (3)$$

Note: These are linear!

Special case: No body force

(2) & (3) $\quad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$

$$\Longrightarrow p(x, \cancel{y}, \cancel{z}, t) = p(x, t)$$

Into eqn(1) & recall: $u(y, z, t)$

$$\underbrace{\rho \frac{\partial u}{\partial t}}_{\text{fct of } y,z,t} = \underbrace{- \frac{\partial p}{\partial x}}_{\text{fct of } x,t} + \mu \underbrace{\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)}_{\text{fct of } y,z,t}$$

This can only be true if (5)

$\frac{\partial p}{\partial x}$ does not actually depend on x!

$$\frac{\partial p}{\partial x} = \underbrace{G(t)}_{\text{some fct of } t \text{ only}}$$

Parallel flow eqns w/o body force

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{1}{\rho} G + \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\ \frac{\partial p}{\partial x} &= G(t) \\ v &= w = 0 \end{aligned}}$$

Example: "Couette flow"

Flow between infinite parallel plates. Upper plane moves to right with veloc U.

fixed

Assume:

- parallel flow in x-direct.
- $u(y, \cancel{x}, \cancel{t}) = u(y)$

  steady

- $\frac{\partial p}{\partial x} = G = 0$ no press. drop.

$$\rho \cancel{\frac{\partial u}{\partial t}} = -\cancel{G} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial t^2}}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{ODE!}$$

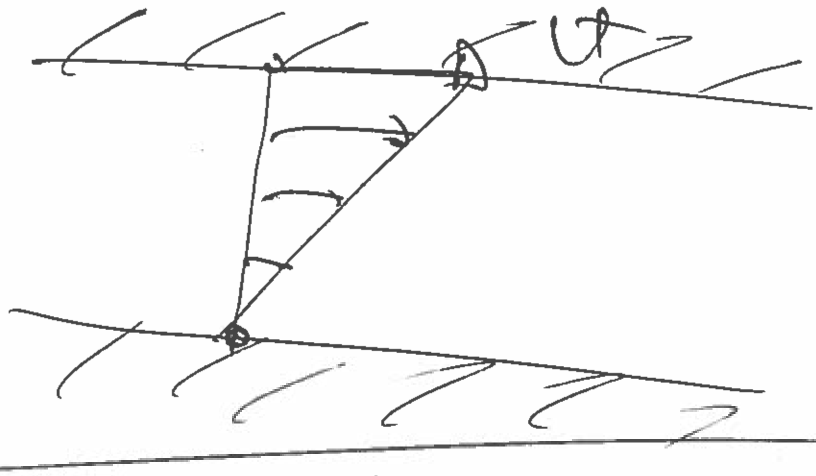
$u(y) = Ay + B$ A, B const

<u>2 BC:</u> No slip:

$$u(y=0) = 0$$
$$u(y=h) = U$$

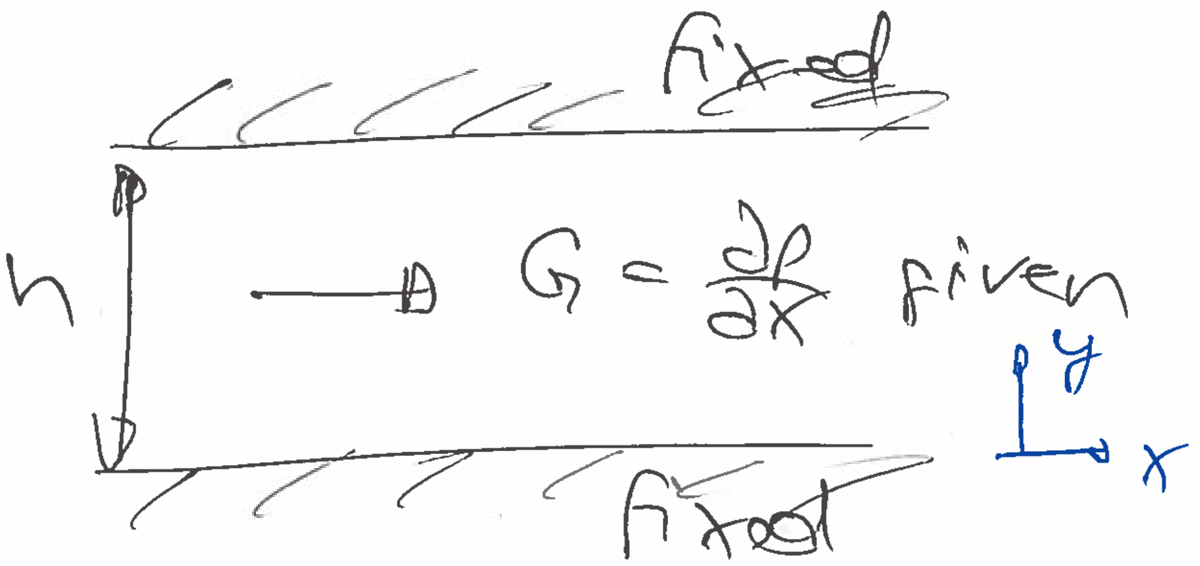
$$\boxed{u(y) = U\frac{y}{h}}$$

linear shear flow



<u>Example:</u> "Poiseuille flow"

pressure driven flow through channel



Assume:

$$u(y, \cancel{z}, \cancel{t}) = u(y)$$

$$\cancel{\rho \frac{\partial u}{\partial t}} = -\underbrace{G(\cancel{t})}_{\text{const.}} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$G = \mu \frac{\partial^2 u}{\partial y^2}$$ 2nd order inhomog. ODE

integrate twice

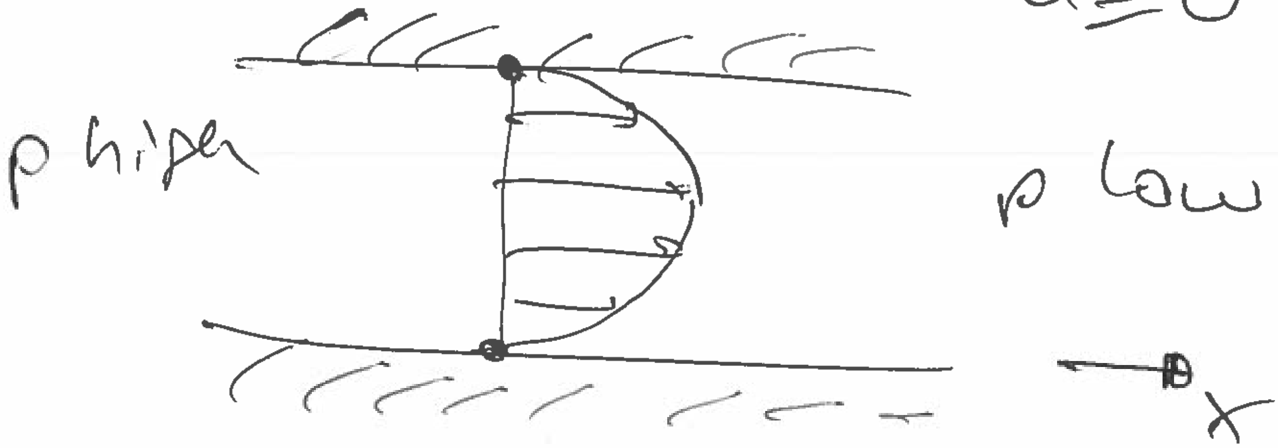
$$u(y) = \frac{1}{2}\frac{G}{\mu} y^2 + Ay + B$$

2 BCs: no slip

$$u(y=0) = 0$$
$$u(y=h) = 0$$

$$\boxed{u(y) = \underbrace{\frac{G}{2\mu}}_{G<0}\underbrace{\left(y^2 - hy\right)}_{<0}}$$

$u \geq 0$



$$G = \frac{dp}{dx} < 0$$ ✓