$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

$$\frac{\partial u_j}{\partial x_j} = 0$$

IC: $\underline{u}(\underline{x}, t=0) = \underline{V}(\underline{x})$ given

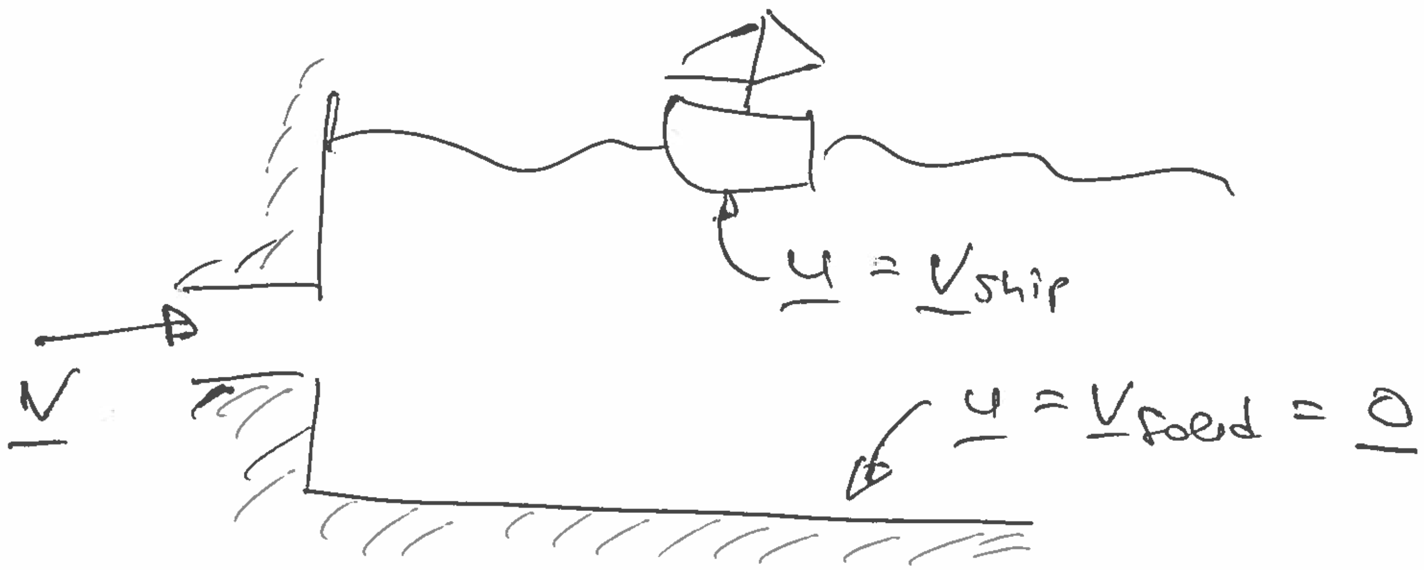
(no IC for pressure!)

BC: Inflow/outflow:

$\underline{u} = \underline{V} \leftarrow$ given

Solid boundaries "no slip":

$\underline{u} = \underline{V}_{solid} \leftarrow$ given



(iii) on free surfaces

We need two conditions:

- Kinematic BC
- traction BC.

(a) Kinematic BC

The posn. of the free surface can always be described implicitly

$$F(x, y, z; t) = 0$$

or

$$f(x, y; t) = 0 \qquad \text{in 2D}$$

At least locally this can always be inverted to

$$z = h(x, y, t)$$

or

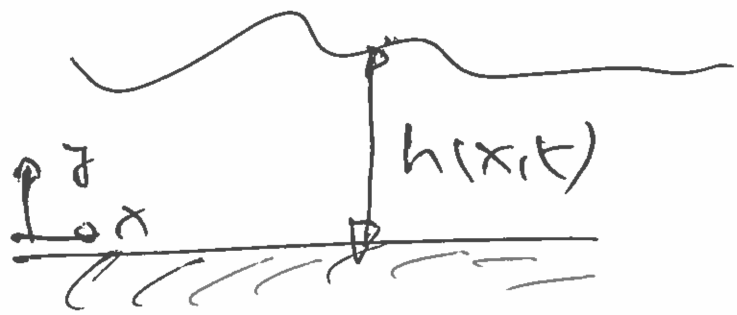
$$y = h(x, t) \qquad \text{in 2D}$$

Observation: Fluid particles which form part of the surface always stay on that surface (exception: cusp)

So for $x_i = x_i^p(t)$: $\boxed{f = 0 \; \forall t}$

$$\Rightarrow \boxed{\frac{Df}{Dt} = 0}.$$

Example:



$$y\big|_{\partial D_F} = h(x,t)$$

we can use

$$f(x,y,t) = h(x,t) - y = 0$$

Kinematic BC $\frac{Df}{Dt}=0$

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}=0$$

$u_1 = u$
$u_2 = v$
$x_1 = x$
$x_2 = y$

$$\frac{\partial f}{\partial t} = \frac{\partial h}{\partial t}$$

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x}$$

$$\frac{\partial f}{\partial y} = -1$$

$$\frac{Df}{Dt} = \boxed{\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}-v=0}$$

<u>Special case 1:</u>

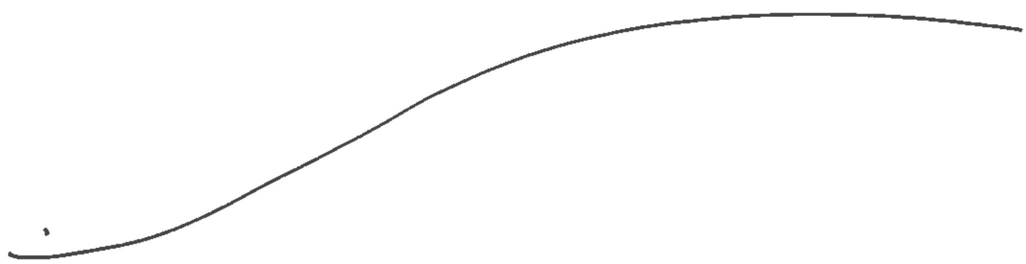
$u=0$ (only vertical flow)

$$\frac{\partial h}{\partial t} = v$$

of course!

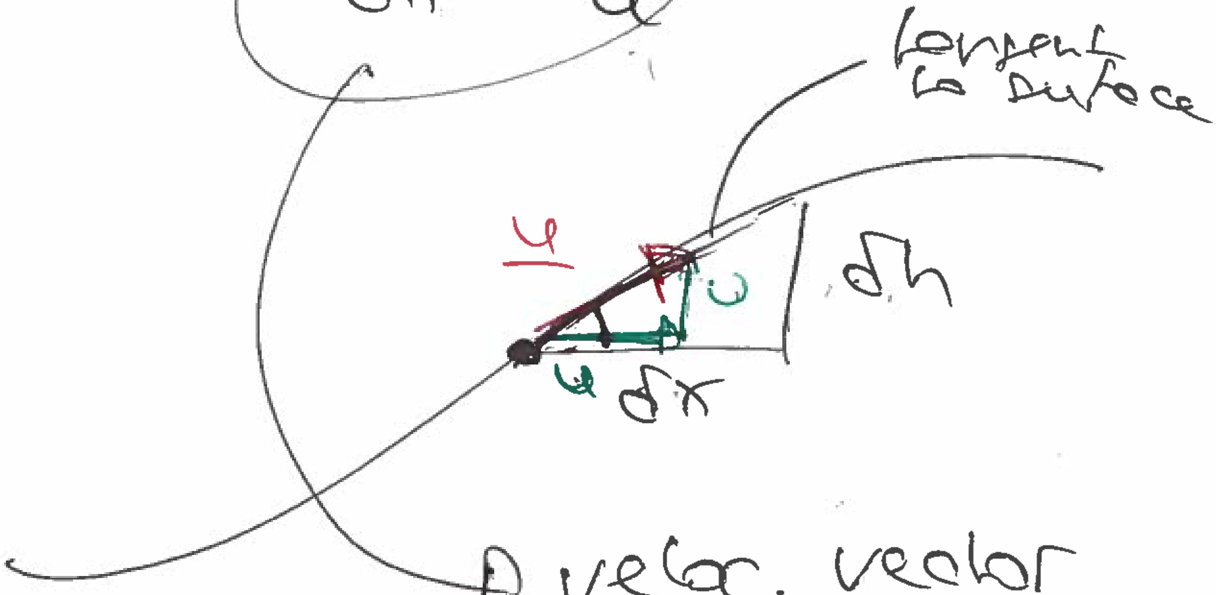
Special case 2:

Fixed free surface posn : $\frac{\partial}{\partial t} = 0$



$$\frac{\partial h}{\partial t} = 0 : \quad u \frac{\partial h}{\partial x} = v$$

$$\frac{\partial h}{\partial x} = \frac{v}{u}$$

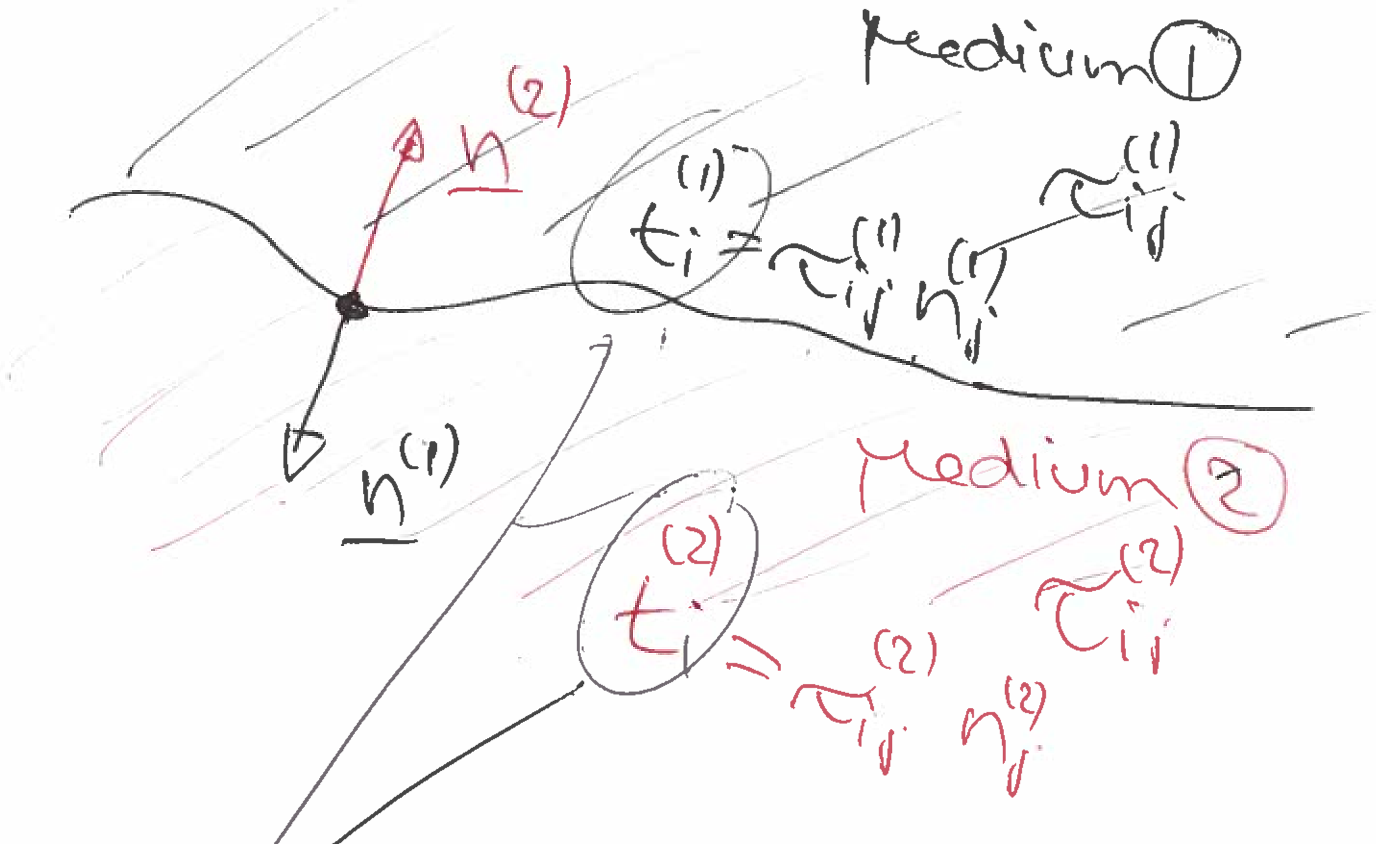


⇒ veloc. vector is tangential to the surface.

of course

(b) traction BC

stress must be continuous across the free surface (apart from surface tension effects)



$t_i^{(1)}$ & $t_i^{(2)}$ are tractions applied onto medium (1) & (2) respectively.

Now: "Action = reaction" (7

$$\underline{t}^{(1)} = -\underline{t}^{(2)}$$

Also: $\underline{n}^{(1)} = -\underline{n}^{(2)}$

So:

$$\boxed{\tau_{ij}^{(1)} n_j = \tau_{ij}^{(2)} n_j}$$

(on the boundary)

for any outer unit normal.

Example Hydrostatic pressure state:

$$\tau_{ij} = -p\,\delta_{ij} \quad :$$

$$\Leftarrow \underbrace{-p^{(1)}\delta_{ij}}_{\tau_{ij}^{(1)}} n_j = \underbrace{-p^{(2)}\delta_{ij}}_{\tau_{ij}^{(2)}} n_j$$

$$-p^{(1)} n_i = -p^{(2)} n_i$$

$$-p^{(1)} \underline{n} = -p^{(2)} \underline{n}$$

$$\underbrace{\left(p^{(1)} - p^{(2)}\right)}_{=0} \underline{n} = \underline{0}$$

$$p^{(1)} = p^{(2)}.$$

of course

In the presence of Surface tension:

$$\tau_{ij}^{(1)} n_j + \sigma \kappa n_i = \tau_{ij}^{(2)} n_j$$

surface tension

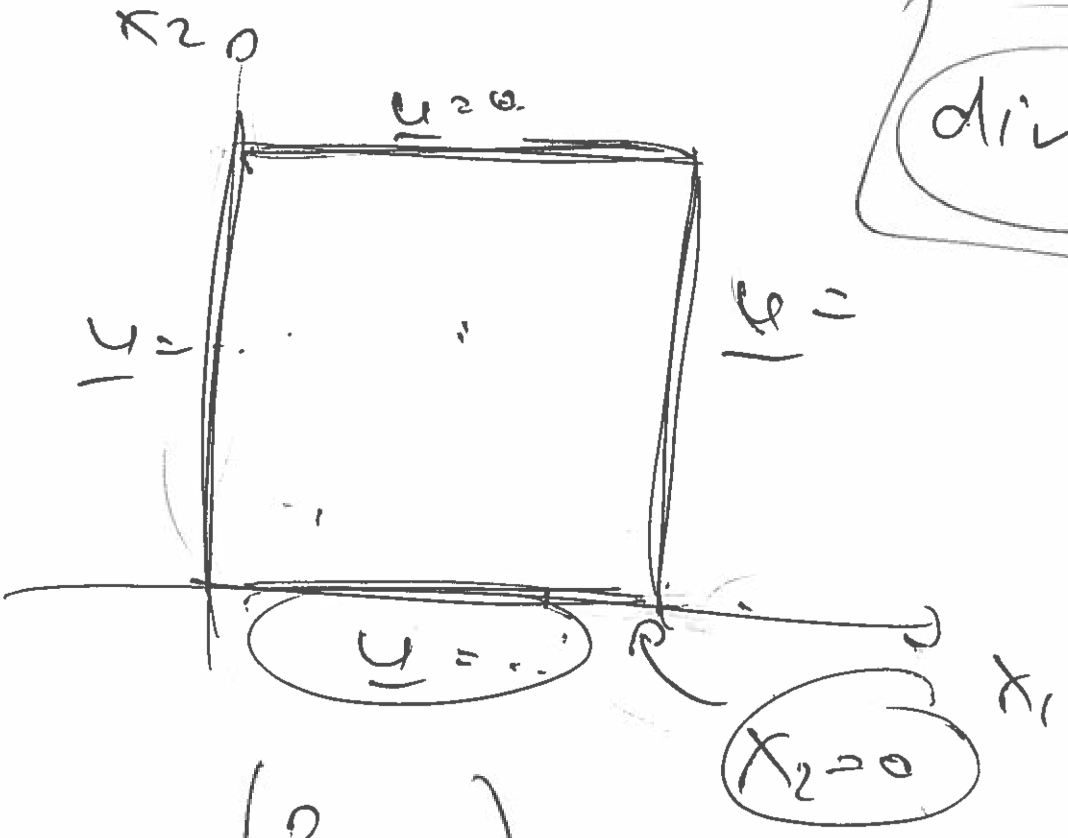
curvature of the surface.

EX. CLASS SCRIBBLES

$$\operatorname{div} \underline{u} = 0$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \Rightarrow A = 1/3$$

$$\operatorname{div} \underline{u} \stackrel{?}{=} 0$$

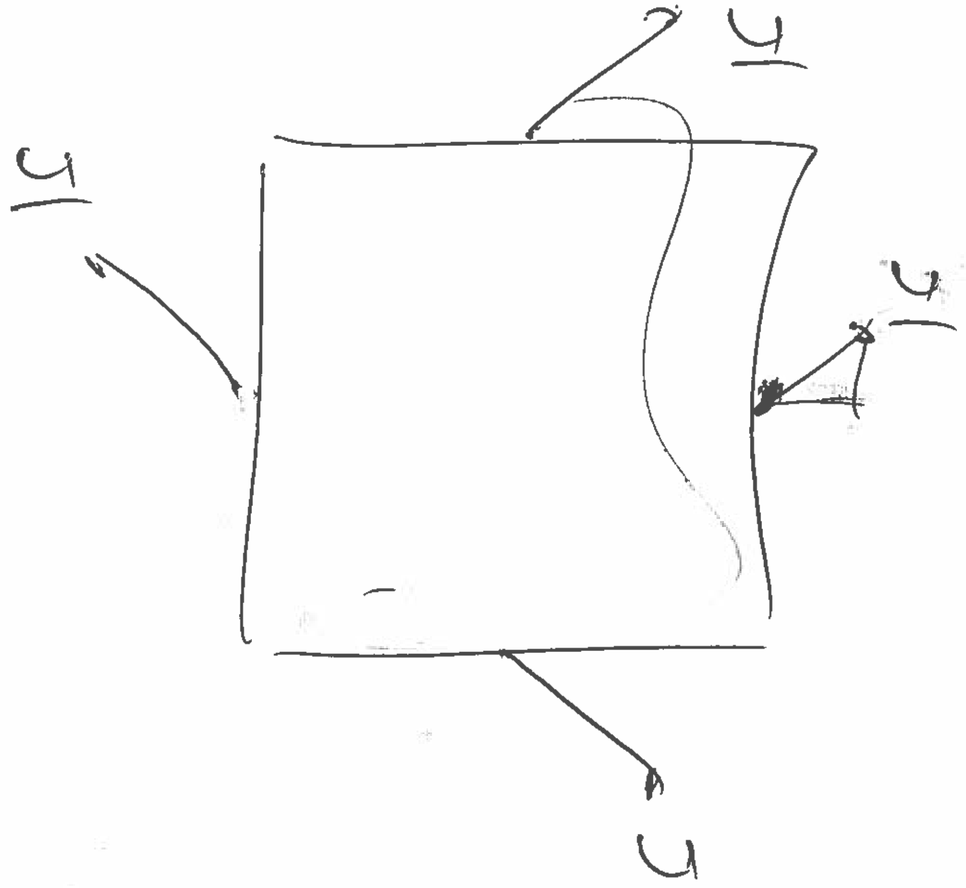


$$\underline{u} = \begin{pmatrix} 3 \\ 4 + 3x_1^2 \end{pmatrix}$$

?

$$\operatorname{div} \underline{u} = \frac{\partial 3}{\partial x_1} + \cancel{\frac{\partial}{\partial x_2}(4 + 3x_1^2)}$$

$$= 0 + 0 = 0$$

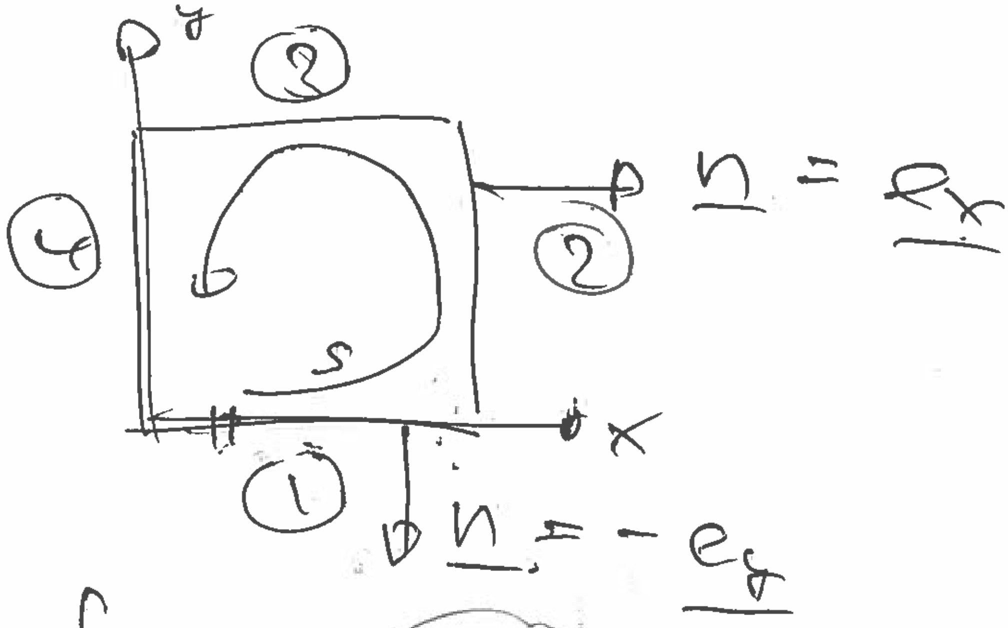


$$\boxed{-\oint\!\!\oint \rho\, \underline{u}\cdot\underline{n}\, dS = \frac{d}{dt}\iiint \rho\, dV}$$

Assume fluid is incompr.

$$\rho \oint\!\!\oint \underline{u}\cdot\underline{n}\, dS = 0$$

$$\boxed{\oint\!\!\oint \underline{u}\cdot\underline{n}\, dS \overset{?}{=} 0}$$



$$\sum_{①}^{④} \int_{①} \underline{u} \cdot \underline{n} \, ds = \begin{cases} 0 \\ \neq 0 \end{cases} \text{or}$$

①: $\underline{u} \cdot \underline{n} = -(4 + 3x_1^2)$

$$\int_{①} \underline{u} \cdot \underline{n} \, ds = \int_{x_1=0}^{1} -(4 + 3x_1^2) \, dx_1$$

②: $$\int \underline{u} \cdot \underline{n} \, ds = \int_{x_2=0}^{1} (3 + x_2 + x_2^2) \, dx_2$$