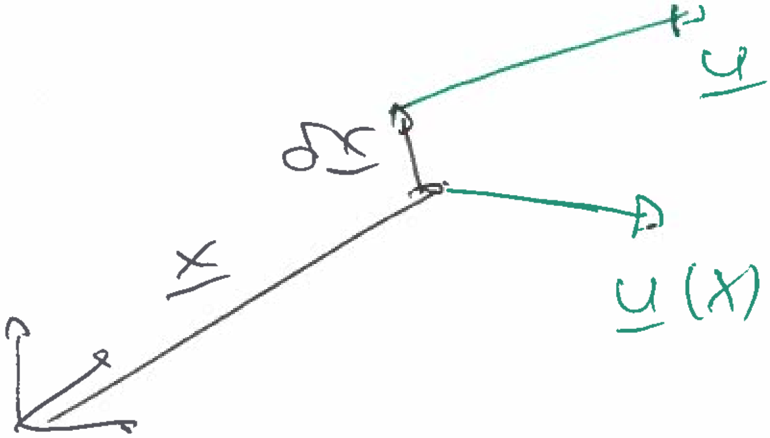


$$\underline{u}(\underline{x}+\delta \underline{x}) = \underline{u}(\underline{x}) + \delta \underline{u}$$

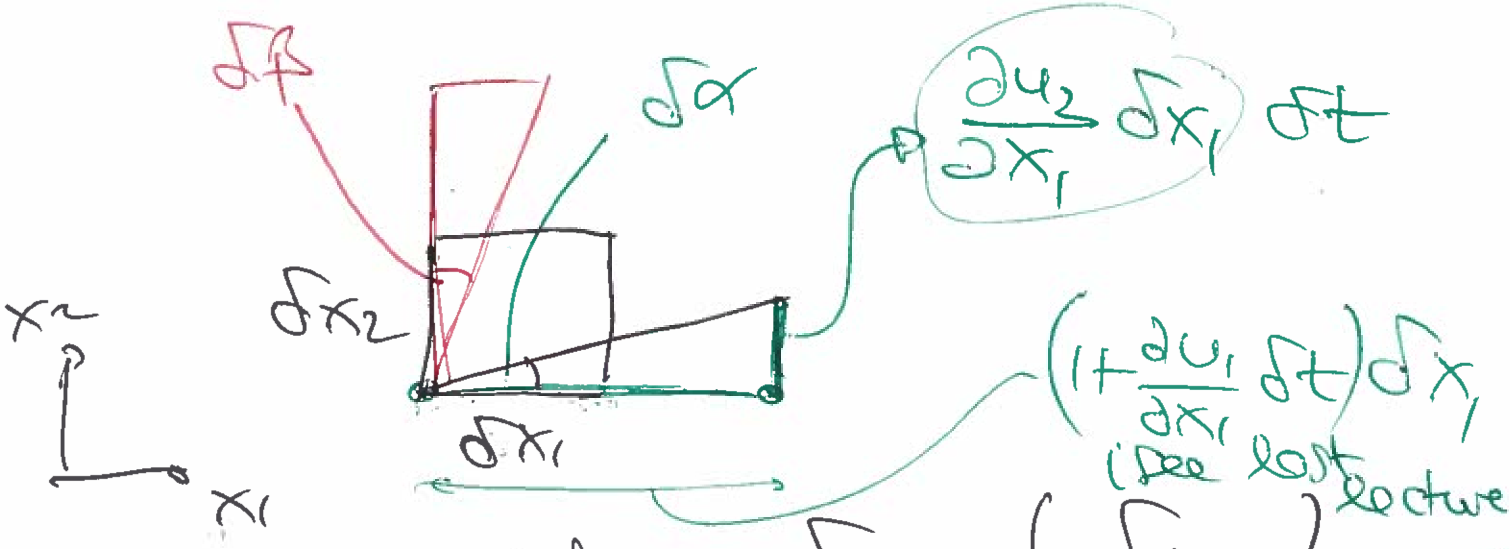
$$\delta u_i = \frac{\partial u_i}{\partial x_j}\,\delta x_j$$

$$\delta u_i = \underbrace{\omega_{ij}}_{\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{rotation}}}\delta x_j + \underbrace{\varepsilon_{ij}}_{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}\delta x_j$$

$e_{(ii)}$ = rate of strain in x_i-direct.

(ii) Shear rate of strain

Illustration (2D)



Compare lines δx_1 (δx_2) after a short time interval δt. In sketch move end point on the left back to original posn.

$$\tan(\delta\alpha) = \frac{\frac{\partial u_2}{\partial x_1}\cancel{\delta x_1}\,\delta t}{\left(1+\frac{\partial u_1}{\partial x_1}\delta t\right)\cancel{\delta x_1}}$$

Now $\delta t \to 0$; $\delta\alpha \to 0$

$$\tan(\delta\alpha) \to \delta\alpha$$

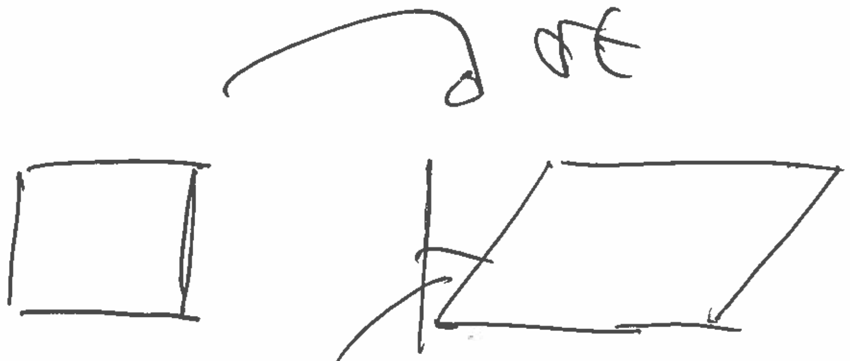
$$\delta\alpha = \frac{\partial u_2}{\partial x_1}\delta t$$

$$\frac{D\alpha}{Dt} = \frac{\partial u_2}{\partial x_1}$$

Similarly:

$$\frac{D\beta}{Dt} = \frac{\partial u_1}{\partial x_2}$$

Now consider the "shear rate", i.e. the rate at which the angle between the two initially orthogonal lines changes:



after suitable rotation.

$$\delta\gamma = \delta\alpha + \delta\beta$$

$$\frac{D\gamma}{Dt} = \frac{D\alpha}{Dt} + \frac{D\beta}{Dt} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}$$

$$= 2\varepsilon_{12}$$

The off diagonal entries in ε_{ij} represent half the shear rate in the x_i, x_j-plane.

Summary:

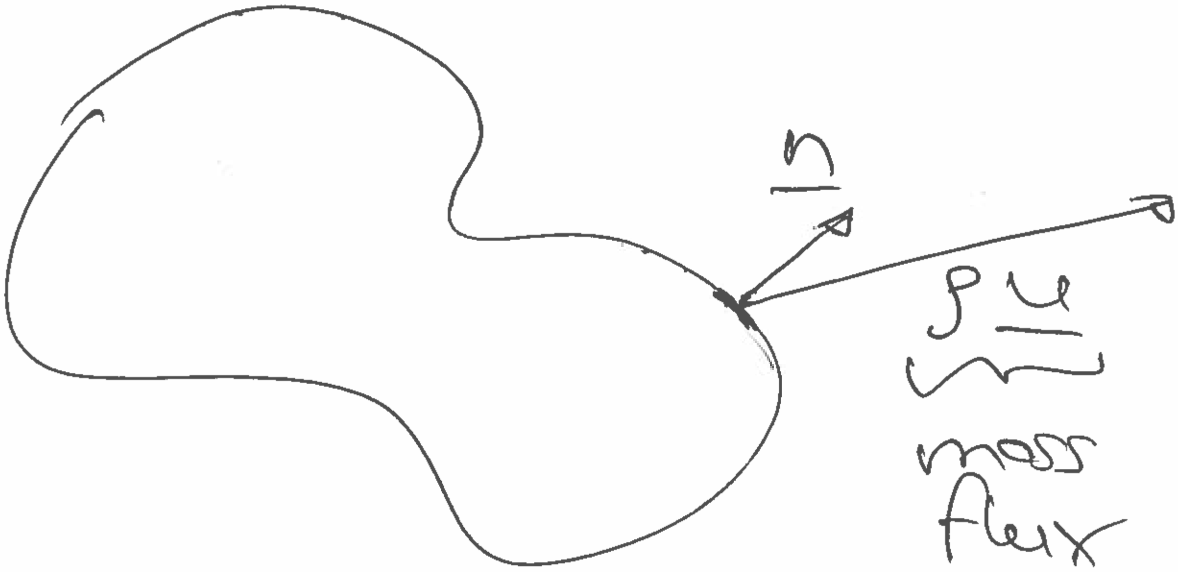
Motion of fluid in the vicinity of a point can be decomposed:

$$\underline{u}(\underline{x}+\delta\underline{x}) = \underbrace{\underbrace{\underline{u}(\underline{x})}_{\text{translation}} + \underbrace{\underline{\Omega}\times\delta\underline{x}}_{\text{rotation}}}_{\text{rigid body}} + \underbrace{\underset{\sim}{E}\,\delta\underline{x}}_{\text{deformation: dilatation \& shearing.}}$$

Equation of continuity

Physics: Mass flux into a spatially fixed (control) volume = rate of change of mass in that volume.

Integral form:



ρ = density $\left[\frac{kg}{m^3}\right]$

$$\boxed{-\oint \rho\, \underline{u}\cdot\underline{n}\, dA = \frac{d}{dt}\int \rho\, dV}$$

$$\underbrace{-\oint \rho\, \underline{u}\cdot\underline{n}\, dA}_{\text{total mass flux over the boundary}} \qquad \underbrace{\frac{d}{dt}\int \rho\, dV}_{\text{mass inside volume}}$$

$$\frac{kg}{\cancel{m^3}}\,\frac{\cancel{m}}{sec}\,\cancel{m^2} \overset{?}{=} \frac{1}{sec}\,\frac{kg}{\cancel{m^3}}\,\cancel{m^3}$$

✓

$$\frac{d}{dt}\int \rho \, dV + \oint (\rho \underline{u}) \cdot \underline{n} \, dA = 0$$

divergence theorem

$$\int \frac{\partial \rho}{\partial t} dV + \int \nabla \cdot (\rho \underline{u}) \, dV = 0$$

$$\int \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) \right] dV = 0$$

For <u>any</u> fixed volume.

$\Rightarrow$ integrand must be zero!

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0}$$

(differential form)

(continuity eqn.)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) = 0$$

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$$\underbrace{\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j}} + \rho \frac{\partial u_j}{\partial x_j} = 0$$

$$\left[\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right]$$

$$\underbrace{\frac{D\rho}{Dt}} + \rho \frac{\partial u_j}{\partial x_j} = 0$$

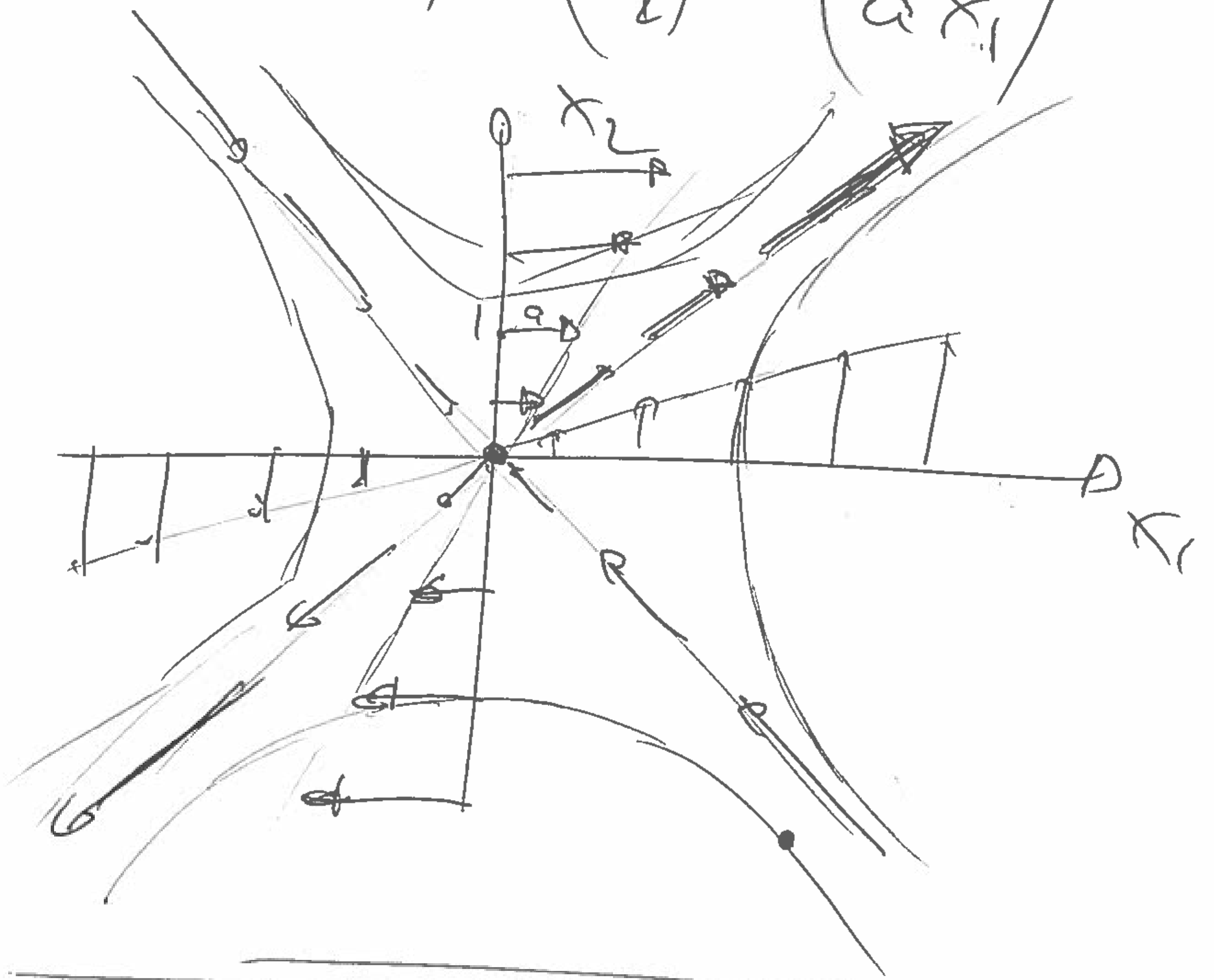
rate of
change of
density of
fluid particle : zero for
<u>incompressible</u>
<u>fluids.</u>

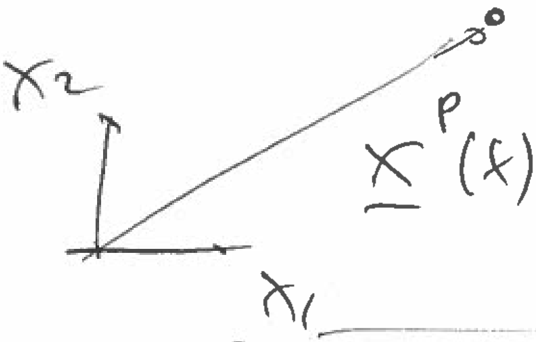
For such fluids the velocity
field has to satisfy

$$\boxed{\frac{\partial u_j}{\partial x_j} = 0} \boxed{= \text{div. } \underline{u}}$$

This is a kinematic constraint!

$$\underline{u}(x_1, x_2) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a x_2 \\ a x_1 \end{pmatrix}$$





$$\boxed{\frac{dx_i^p}{dt} = u_i(x_1^p(t), x_2^p(t))}$$

$$\frac{dx_1^p}{dt} = a x_2^p \qquad x_1^p(t=0) = X_1$$

$$\frac{dx_2^p}{dt} = a x_1^p \qquad x_2^p(t=0) = X_2$$

$$\frac{d^2x_2}{dt^2} = a\frac{dx_1}{dt} = \underbrace{a^2 x_2}_{\text{from } 1^{st} \text{ ODE}}$$

$$\frac{d^2x_2}{dt^2} - a^2 x_2 = 0$$

$$x_2(x) = A e^{at} + B e^{-at}$$

<u>IC:</u> $x_2(t=0) = X_2$

$$\left.\frac{dx_2}{dt}\right|_{t=0} = a x_1\Big|_{t=0} = a X_1$$