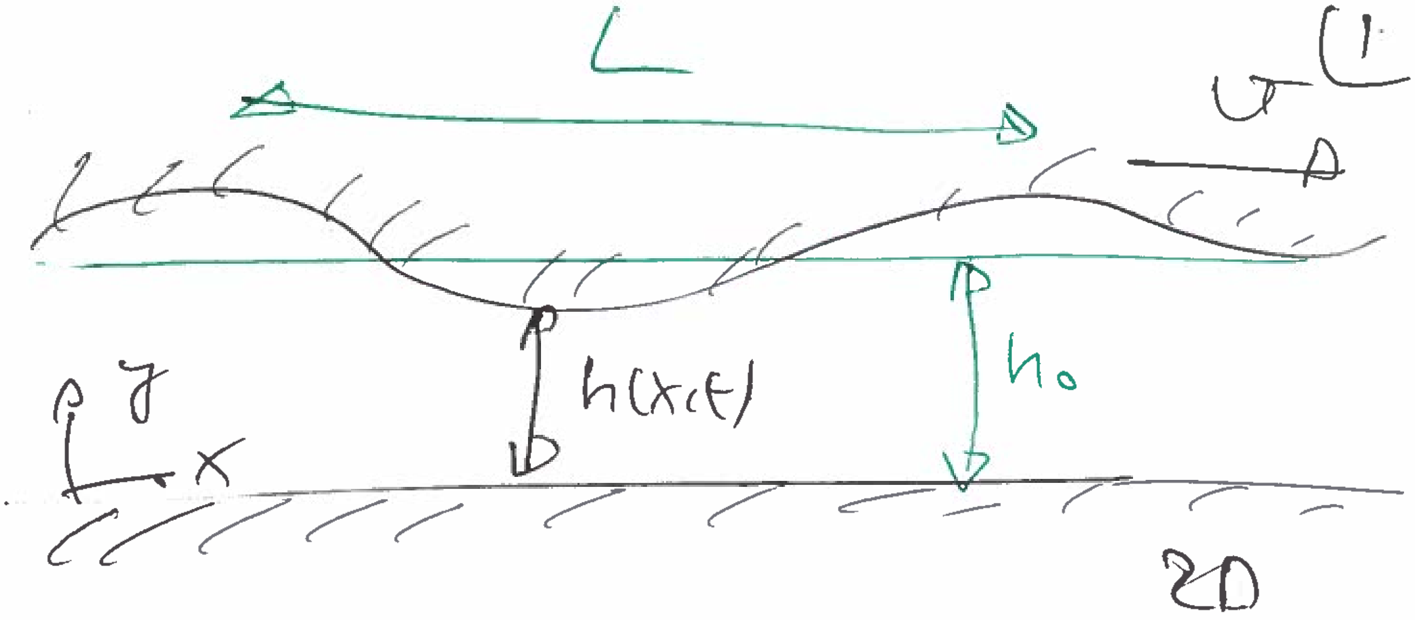


Narrow, gently varying gap width

$$h_0 \ll L$$

Scale:

$x = L\tilde{x}$ $\qquad$ $t = \frac{L}{U}\tilde{t}$

$y = h_0\tilde{y}$ $\qquad$ $p = P\tilde{p}$

$u = U\tilde{u}$

$v = V\tilde{v}$ $\qquad$ V, P unknown

$$\frac{h_0}{L} \ll 1$$

Continuity:

$$\frac{U}{L}\frac{\partial \tilde u}{\partial \tilde x} + \frac{V}{h_0}\frac{\partial \tilde v}{\partial \tilde y} = 0$$

These terms have to balance (otherwise we get only trivial results):

$$\frac{U}{L} = \frac{V}{h_0} \Rightarrow V = U \underbrace{\frac{h_0}{L}}_{\ll 1}$$

into x-comp. of mom. eqns:

$$\rho\left(\frac{U^2}{L}\frac{\partial \tilde u}{\partial \tilde t} + \frac{U^2}{L}\tilde u\frac{\partial \tilde u}{\partial \tilde x} + \frac{U^2 \cancel{h_0}/L}{\cancel{h_0}}\tilde v\frac{\partial \tilde u}{\partial \tilde y}\right) =$$

$$-\frac{P}{L}\frac{\partial \tilde p}{\partial \tilde x} + \mu\left(\frac{U}{L^2}\frac{\partial^2 \tilde u}{\partial \tilde x^2} + \frac{U}{h_0^2}\frac{\partial^2 \tilde u}{\partial \tilde y^2}\right)$$

$$\rho\frac{U^2}{L}\frac{D\tilde u}{D\tilde t} = -\frac{P}{L}\frac{\partial \tilde p}{\partial \tilde x} + \frac{\mu U}{h_0^2}\left(\cancel{\underbrace{\left(\frac{h_0}{L}\right)^2}_{\ll 1}\frac{\partial^2 \tilde u}{\partial \tilde x^2}} + \frac{\partial^2 \tilde u}{\partial \tilde y^2}\right)$$

$$\cancel{\underbrace{\underbrace{\frac{\rho U h_0}{\mu}}_{Re}\underbrace{\left(\frac{h_0}{L}\right)}_{\ll 1}}\frac{\partial \tilde{u}}{\partial \tilde{t}}} = -\frac{P}{\left(\frac{\mu U}{h_0}\right)}\left(\frac{h_0}{L}\right)\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

small if product of Re (formed with h_0) & $\frac{h_0}{L}$ is small.

Balance of the two remaining terms provides the pressure scale:

$$P = \frac{\mu U}{h_0}\frac{1}{\left(h_0/L\right)}$$

in that case the nondim. eqns become

$$0 = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

or back to dimensional form:

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$

EXERCISE: repeat for the y-component of the mom. eqn. (No undetermined scales left!)

$$\frac{\partial \tilde{p}}{\partial \tilde{y}} = 0 \quad \text{or} \quad \frac{\partial p}{\partial y} = 0$$

So, the governing equations are:

$$\boxed{\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \qquad (1) \\ \frac{\partial p}{\partial y} &= 0 \qquad (2) \end{aligned}}$$

Assuming that $\frac{h_0}{L} \ll 1$ and

$$\underbrace{\frac{\rho U h_0}{\mu}}_{Re} \left(\frac{h_0}{L}\right) \ll 1$$

Note: These are the parallel flow eqns without body force.

Locally the flow behaves [5
as if it was in an
infinitely long channel of
the local width.

BC: $u(y=0) = 0$

$u(y = h(x,t)) = U$

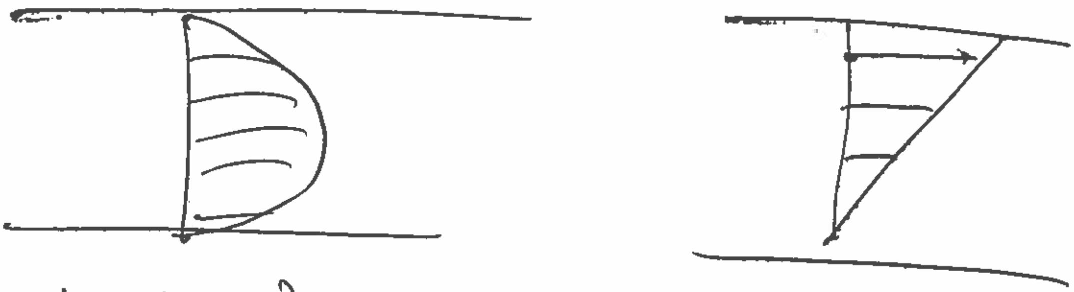
(1) $\underbrace{\frac{\partial p}{\partial x}}_{\text{does not depend on } y;\text{ see (2)}} = \mu \frac{\partial^2 u}{\partial y^2}$

integrate twice:

$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + Ay + B$

Apply BCs:

$$\boxed{u(x,y,t) = \underbrace{\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h(x,t)\,y\right)}_{\text{pressure driven flow}} + \underbrace{U\frac{y}{h(x,t)}}_{\text{shear flow}}}$$

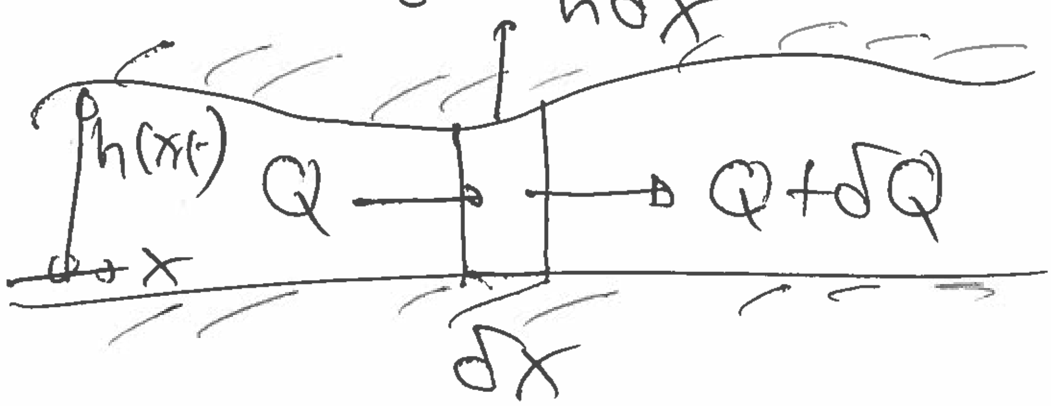


but: what is the pressure?

Additional requirement:
volume conservation.

Def. local volume flux:

$$Q(x) = \int_0^h u \, dy$$



$$\delta Q + \dot{h}\,\delta x = 0$$

Take limit $\delta x \to 0$

$$\frac{\partial q}{\partial x} = -\dot{h} = -\frac{\partial h}{\partial t}$$