Re << 1

$$\nabla^4 \psi = 0 \quad + \text{BC}$$

$$\underline{u}(r,\varphi) = \underline{u}(\varphi) \; !$$

Discussion:

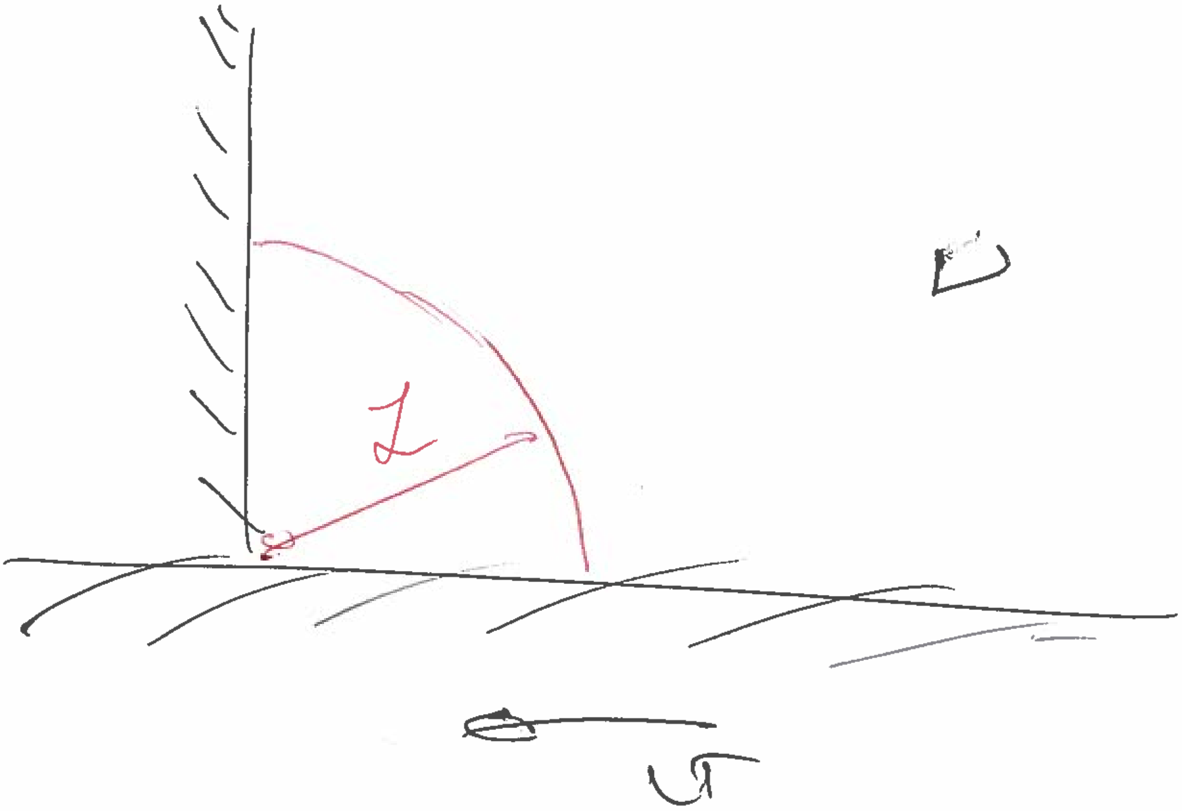
(I) Nonuniformity of the soln

$$Re = \frac{U\mathcal{L}}{\nu} << 1$$

Veloc. scale: U
Viscosity: ν
What is the lengthscale: $\mathcal{L}$?

This problem does not have an intrinsic lengthscale!

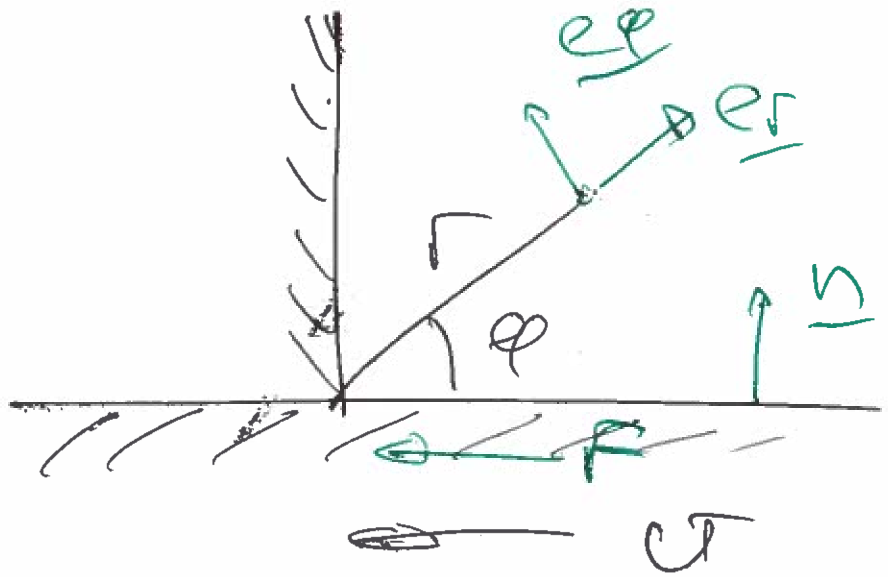
We have to choose a distance L (measured from the corner) over which we are interested in the flow.



The Reynolds number formed with this lengthscale gives an indication how valid the Stokes eqns are as an approx. to the full NSE eqns.

The reverse also holds:
At large distances from the corner the Stokes eqns are not valid.

II Traction on the bottom wall



Force: we want the tangential component of the force acting onto the plate

$$F = \left| \int_0^\infty \tau_{r\varphi}\Big|_{\varphi=0} dr \right|$$

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu\, e_{ij}$$

$$\{i,j\} = \{r,\varphi\}$$

$$\tau_{r\varphi} = 2\mu \underbrace{e_{r\varphi}}_{\text{see handout}}$$

$$\tau_{r\varphi} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \varphi}\right)$$

at $\varphi = 0$

Recall: u, v do not depend on r!

$$r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) \sim r\, r^{-2} \sim \frac{1}{r}$$

$$\frac{1}{r}\frac{\partial u}{\partial \varphi} \sim \frac{1}{r}$$

$$\Rightarrow \tau_{r\varphi} \sim \frac{1}{r}$$

$$F \sim \mu \int_0^\infty \frac{1}{r}\, dr \longrightarrow \infty$$

diverges

$\Rightarrow$ need an ∞ large force to move the plate.

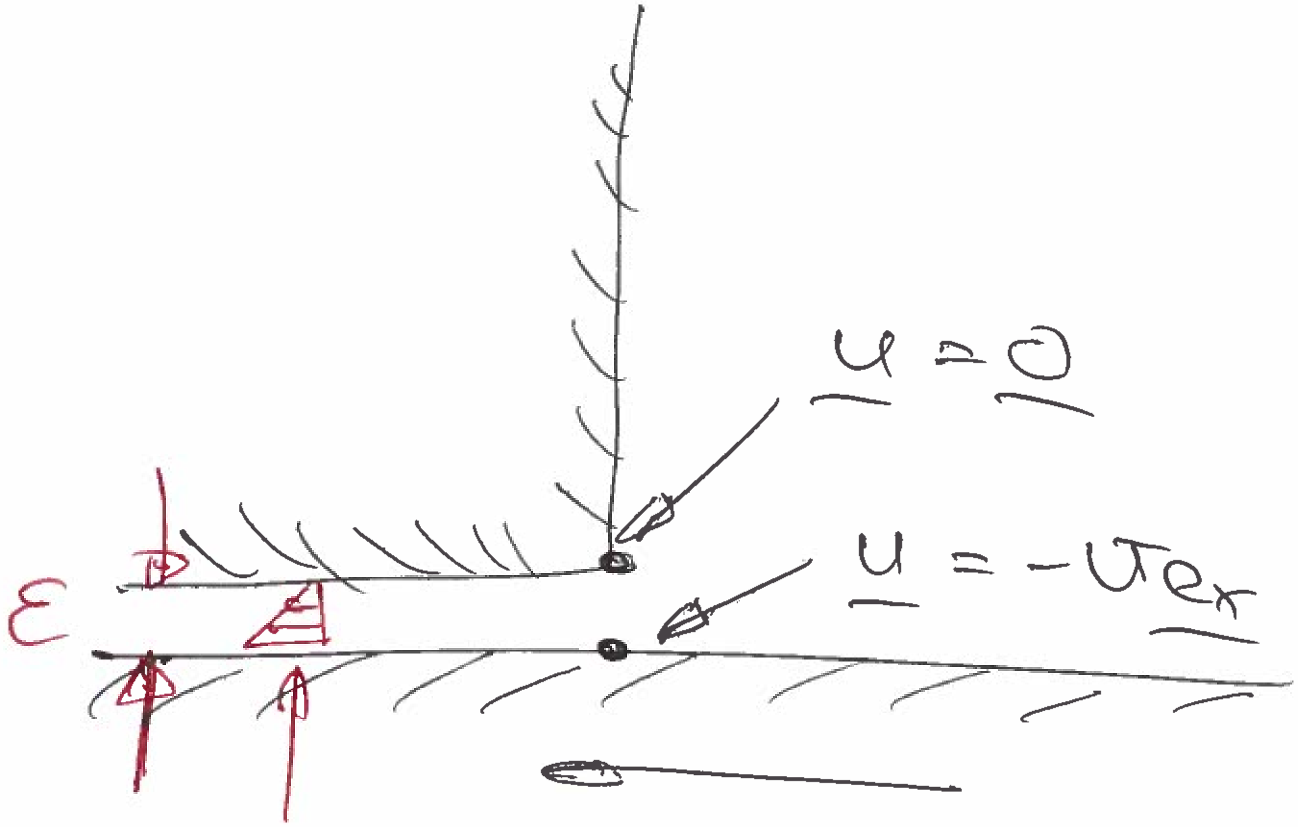
REALLY ?

<u>Note:</u> This is partly due to the infinite size of the plate.

If instead we compute force on a finite part of the plate:

$$F_L \sim \mu \int_0^L \frac{1}{r}\, dr$$ is still infinite

<u>Note:</u> The problem arises at $r = 0$.



Couette flow:
veloc. varies linearly
between 0 and U

$\Rightarrow$ the rate of strain is

$$\frac{U}{\varepsilon} \to \infty \quad \text{as} \quad \varepsilon \to 0$$

Infinite shear rate $\to$ infinite shear stress.

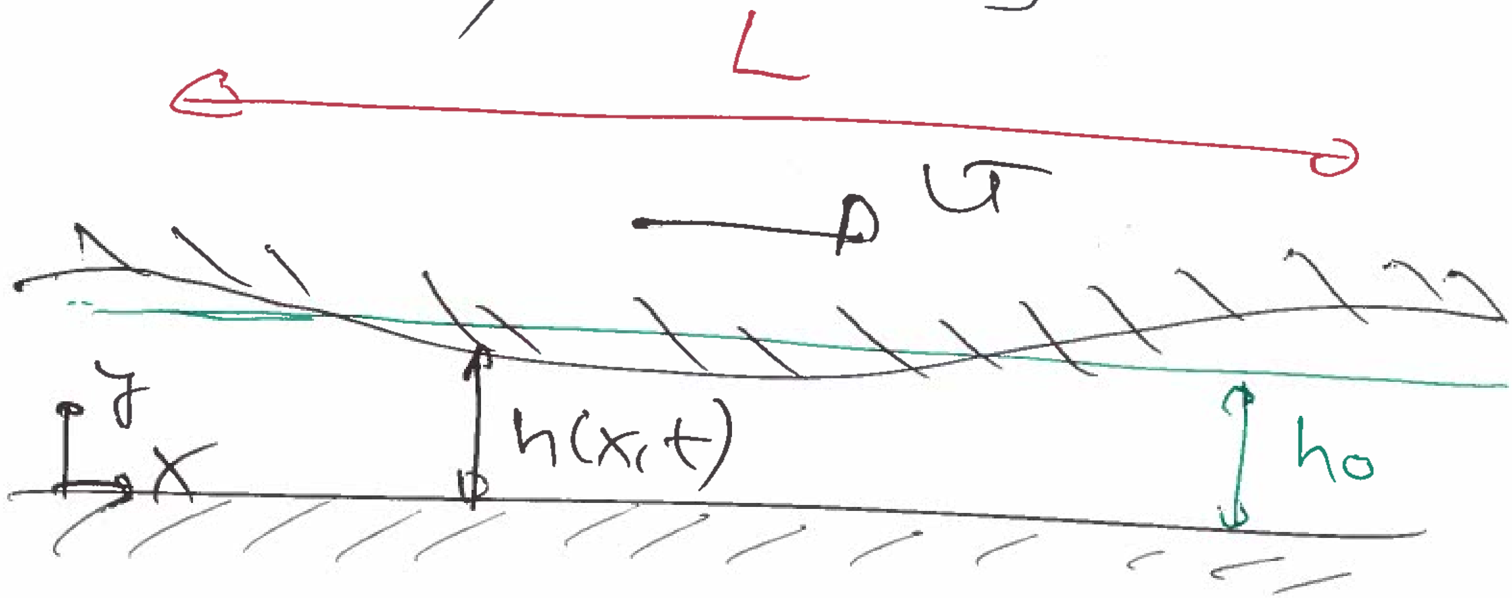
But N.-St. eqns are only valid up to lengthscales ~~F~~ on which molecular effects are irrelevant.

Beyond these different physical effects come into play & "regularise" the problem.

§... Lubrication theory

(Another example of scaling)

2D



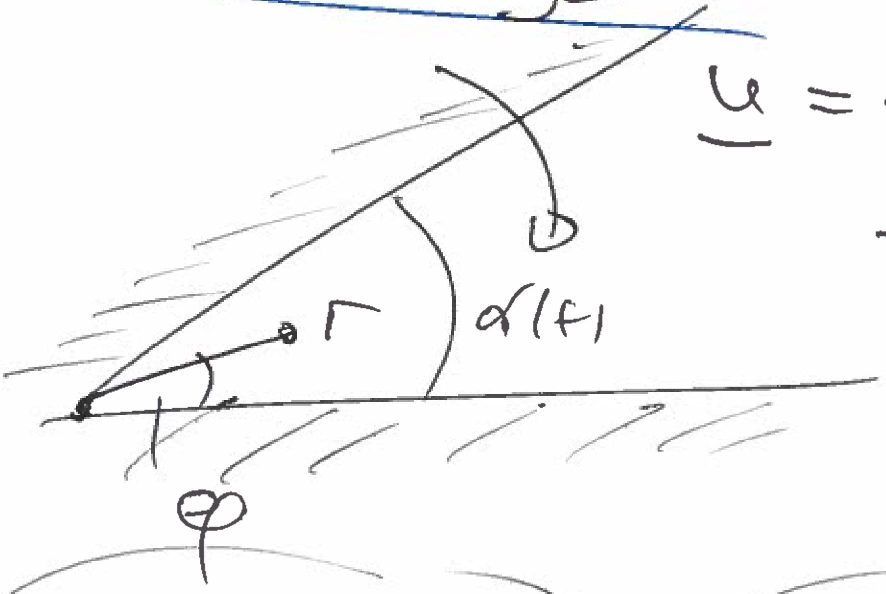
Gap is very narrow:

$$h_0 \ll L$$

typical gap width

length-scale over which the gap width changes

Ex. class



$$\underline{u} = -\Omega r \underline{e_\varphi} + 0\,\underline{e_r}$$

$$\underline{u} = \underline{0}$$

$$u = u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \qquad v = u_\varphi = -\frac{\partial \psi}{\partial r}$$

quasi-steady

$$\nabla^4 \psi = 0$$

$\underline{\varphi = 0}$: $\boxed{\frac{1}{r}\frac{\partial \psi}{\partial \varphi} = 0 \text{ for } \varphi = 0 \ \forall r}$

$$-\frac{\partial \psi}{\partial r} = 0 \text{ for } \varphi = 0 \ \forall r$$

$$\rightarrow \boxed{\psi\big|_{\varphi=0} = C_1} \quad (*)$$

$\varphi = \alpha(r)$:

$$\boxed{\frac{1}{r}\frac{\partial \psi}{\partial \varphi} = 0 \quad \text{for } \varphi = \alpha \ \forall r}$$

$$-\frac{\partial \psi}{\partial r} = -\Omega r \quad \text{for } \varphi = \alpha \ \forall r$$

$$\boxed{\psi\Big|_{\varphi=\alpha} = \frac{1}{2}\Omega r^2 + C_2 \ (**)}$$

Compare $(*)$ & $(**)$:
Both apply at $r = 0$ & there is no flow into the corner $\Rightarrow$ ψ has to be continuous:

$$C_1 = C_2 = C = 0$$

$$\nabla^4 \psi = 0 \qquad \text{lin, hom}$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \varphi}\bigg|_{\varphi=0} = 0 \qquad \text{lin hom}$$

$$\psi\big|_{\varphi=0} = 0 \qquad \text{lin hom}$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \varphi}\bigg|_{\varphi=\alpha} = 0 \qquad \text{lin hom}$$

$$\psi\big|_{\varphi=\alpha} = \frac{1}{2}\Omega r^2 \qquad \text{lin inhomog.}$$

$$\psi(r, \varphi; \alpha, \Omega)$$

$$\psi = \Omega f(r, \varphi; \alpha)$$

Dimension

$$[\psi] = \left\{ \frac{m^2}{sec} \right.$$

$$u = \frac{\partial \psi}{\partial y}$$

$$\frac{m}{sec} = \frac{[\psi]}{m}$$

(3

4

$$[\Omega] = \frac{1}{\sec}$$

f must have dimensions m^2

$$\psi = \underbrace{\Omega r^2 \hat{f}(r, \varphi; \alpha)}_{f}$$

$\hat{f}$ must be a nondim fct of its arguments

r is the only quantity with dimension m

$\Longrightarrow$ $\hat{f}$ cannot depend on r

$$\psi = \Omega r^2 \hat{f}(\varphi; \alpha)$$

$$\psi(r, \varphi) = \Omega r^2 \hat{f}(\varphi)$$

into PDE $\longrightarrow$ ODE.