1

$$\boxed{\begin{gathered} Re \frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2} \\ \frac{\partial \tilde{u}_j}{\partial \tilde{x}_j} = 0 \end{gathered}}$$

$$Re = \frac{UL\rho}{\mu} \to 0$$

$$0 = -\nabla p + \mu \nabla^2 \underline{u}$$

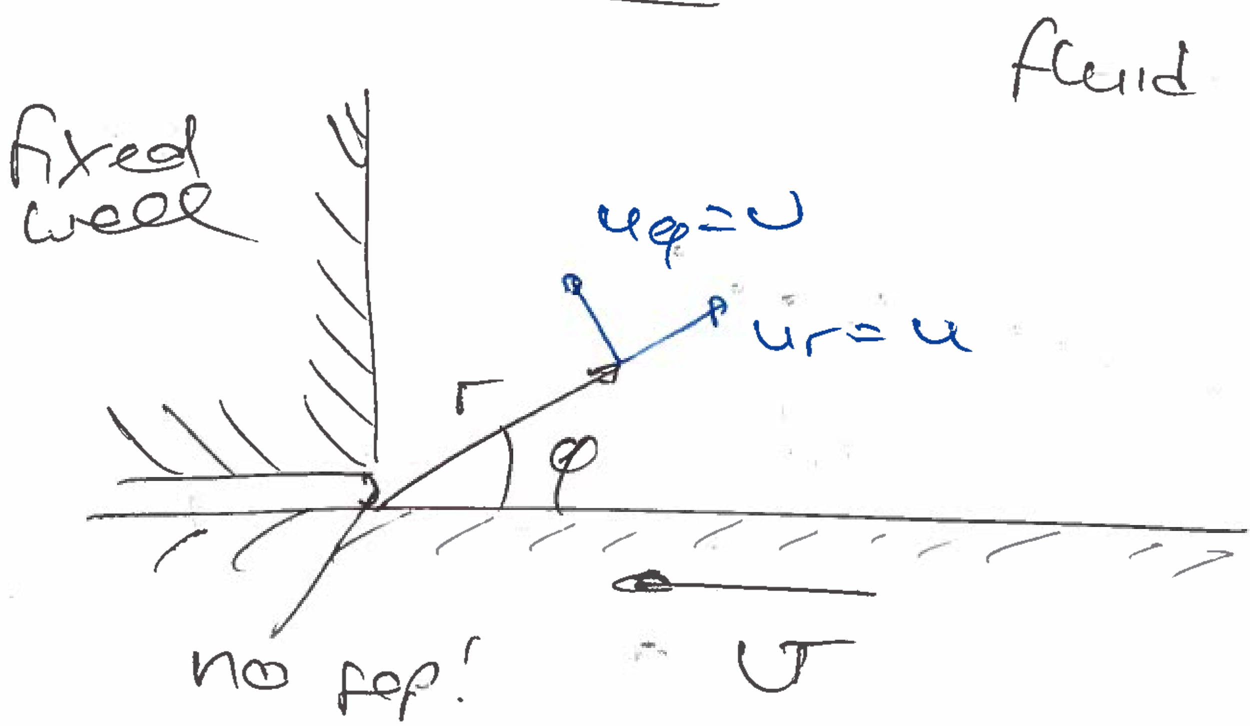
$$\nabla \cdot \underline{u} = 0$$

<u>2D:</u>

$$\boxed{\begin{gathered} u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \\ \nabla^4 \psi = 0 \end{gathered}}$$

Stokes flow example

Scraping flow



Assume: slow, steady viscous flow

$\Rightarrow$ Stokes eqns apply because

$Re \ll 1$.

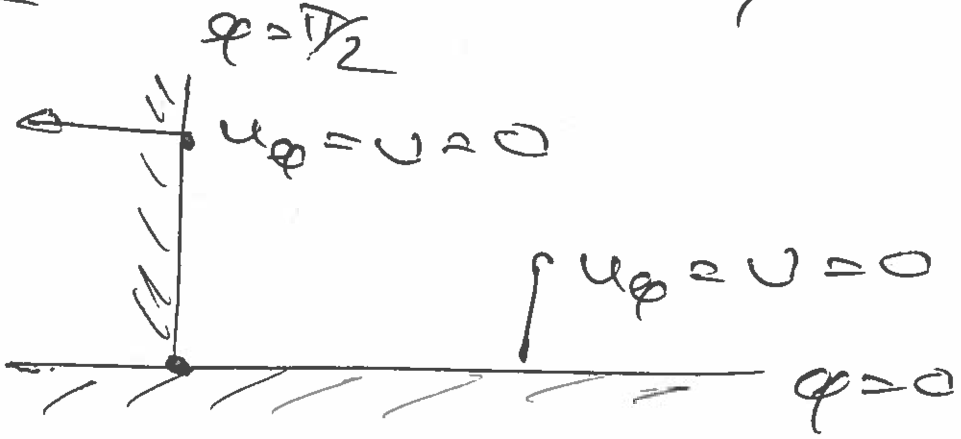
$$\nabla^4 \psi = 0$$

use polar coords.

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

$$u = u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \; ; \quad v = u_\varphi = -\frac{\partial \psi}{\partial r}$$

BC: Impermeability:



$$v = -\frac{\partial \psi}{\partial r} = 0 \quad \text{at } \varphi = 0 \quad \forall r$$

$$\psi = \text{const} = C_1 \quad \text{at } \varphi = 0$$

$$v = -\frac{\partial \psi}{\partial r} = 0 \quad \text{at } \varphi = \frac{\pi}{2} \quad \forall r$$

$$\psi = \text{const} = C_2 \quad \text{at } \varphi = \frac{\pi}{2}$$

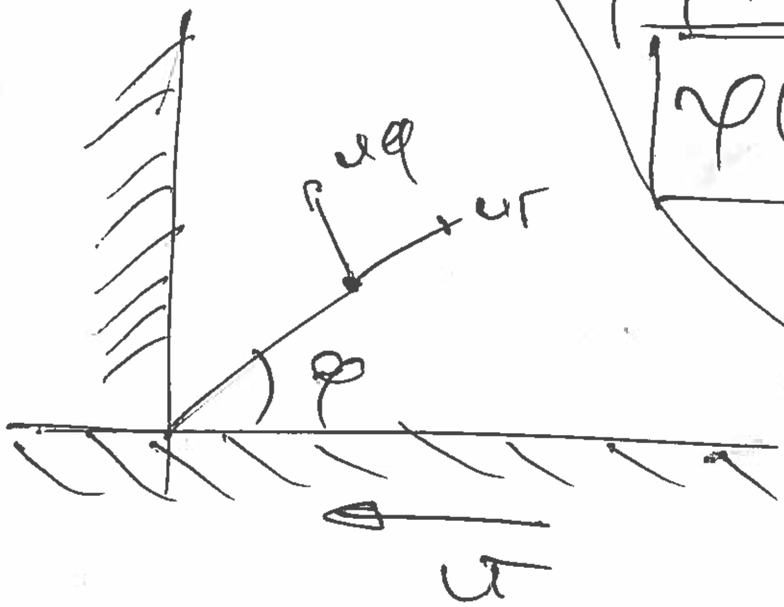
At $r=0$: both BCs apply & the streamfct has to be continuous (otherwise there would be a finite flow into the corner): $C_1 = C_2 = C$

C is an arbitrary constant; value does not affect the veloc. $\Rightarrow$ set $C = 0$.

<u>No slip:</u>

$$\boxed{\psi(\varphi = 0) = 0} \quad (1)$$

$$\boxed{\psi\left(\varphi = \frac{\pi}{2}\right) = 0} \quad (2)$$



$$u = u_r = \boxed{\frac{1}{r}\frac{\partial \psi}{\partial \varphi} = 0 \quad \text{at } \varphi = \frac{\pi}{2}} \quad (3)$$

$$u = u_r = \boxed{\frac{1}{r}\frac{\partial \psi}{\partial \varphi} = -U \quad \text{at } \varphi = 0} \quad (4)$$

<u>Note:</u> soln must be linear in U!

Try to find a separated soln.

$$\psi(r,\varphi) = g(r)\, f(\varphi)$$

Note: BCs (3) & (4) show that

$\frac{1}{r}\frac{\partial \psi}{\partial \varphi}$ should be indep. of r (at 2 values of φ)

Try: $g(r) = Ur$

$$\boxed{\psi(r,\varphi) = Ur\, f(\varphi)}$$

BCs:

$\varphi = 0: \psi = 0 \Rightarrow f(0) = 0$

$\varphi = \pi/2: \psi = 0 \Rightarrow f(\frac{\pi}{2}) = 0$

$\varphi = 0: \frac{1}{r}\frac{\partial \psi}{\partial \varphi} = -U \Rightarrow f'(0) = -1$

$\varphi = \frac{\pi}{2}: \frac{1}{r}\frac{\partial \psi}{\partial \varphi} = 0 \Rightarrow f'(\frac{\pi}{2}) = 0$

PDE: $\nabla^4\psi = \nabla^2\nabla^2\psi = 0$ (6

$$\nabla^2\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right) U r f(\varphi)$$

$$= \frac{U}{r} f + \frac{U}{r} f'' = U r^{-1}(f + f'')$$

$$\nabla^4\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right) U r^{-1}(f + f''$$

$$\boxed{\nabla^4\psi = \frac{U}{r^3}\underbrace{\left(f + 2f'' + f^{IV}\right)}_{=0} = 0}$$

$$f(\varphi) \sim e^{\lambda\varphi}$$

char. poly:

$$1 + 2\lambda^2 + \lambda^4 = 0$$

$$(\lambda^2 + 1)^2 = 0$$

$\lambda_{1234} = \pm i$ purely complex

$$f(\varphi) = A\sin\varphi + B\cos\varphi + C\varphi\sin\varphi + D\varphi\cos\varphi$$

Now apply 4BCs for A, B, C, D

$$\boxed{\psi = \frac{Ur}{\left(\frac{\pi}{2}\right)^2 - 1}\left(-\left(\frac{\pi}{2}\right)^2\sin\varphi + \varphi\cos\varphi + \frac{\pi}{2}\varphi\sin\varphi\right).}$$

So:

$u_\varphi = v = -\frac{\partial\psi}{\partial r}$ indep. of r!

$u_r = u = \frac{1}{r}\frac{\partial\psi}{\partial\varphi}$ also indep. of r!!

(8